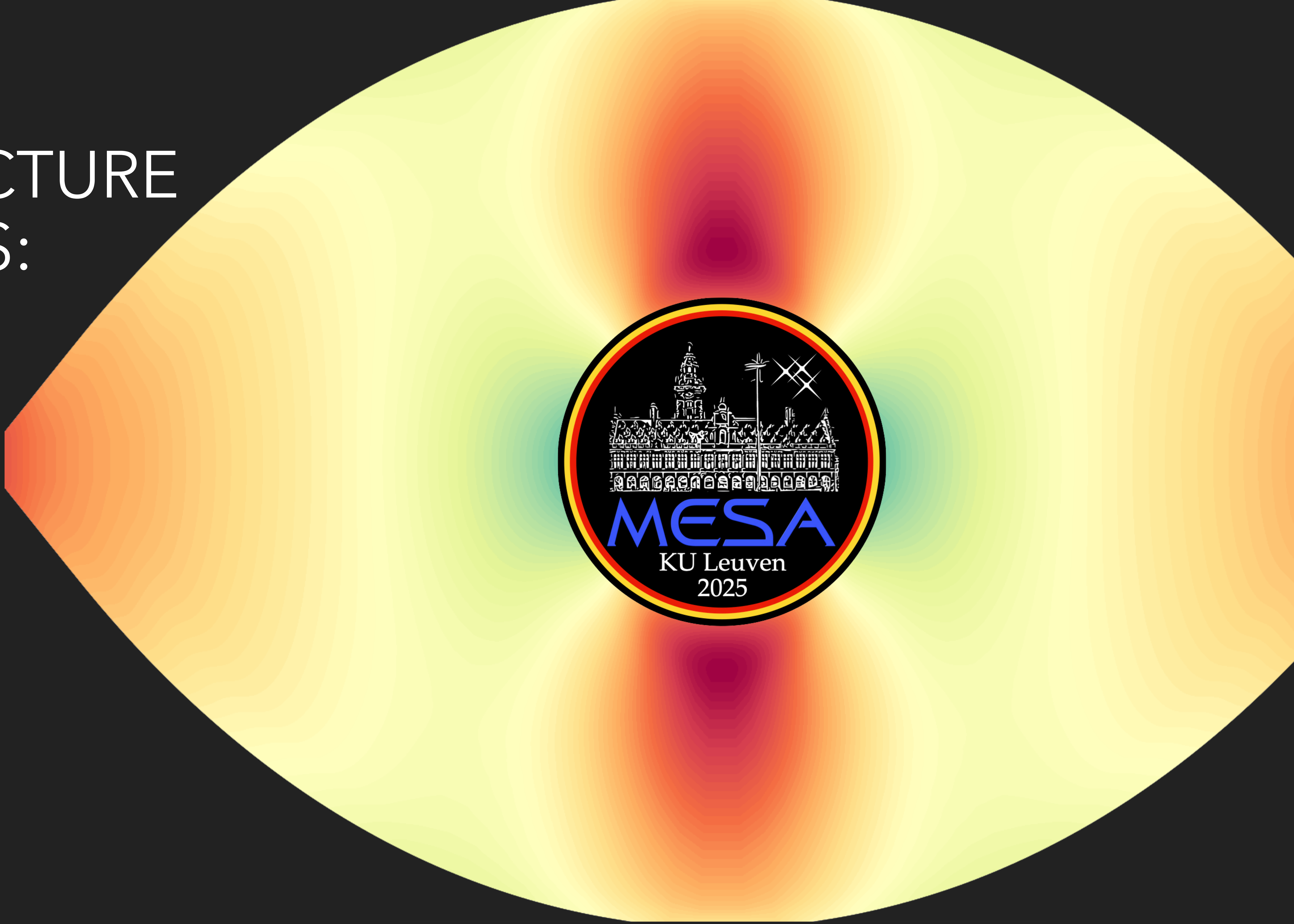
ROTATING STELLAR STRUCTURE AND EVOLUTION MODELS: 1D VS 2D

MESA school 2025, Day 2



TAs:
Beatriz Bordadágua
Philip Mocz
Tryston Raecke

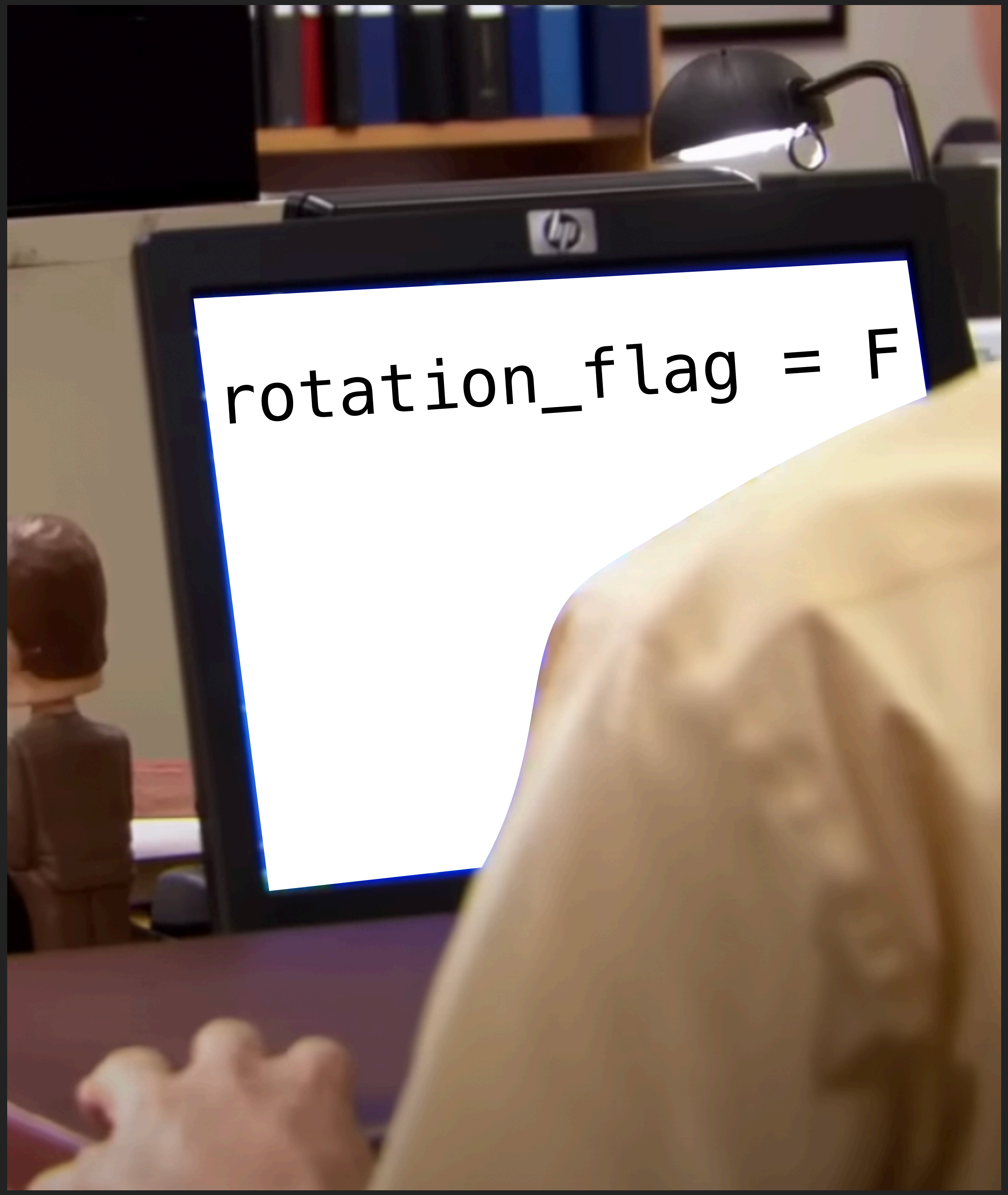
joey.mombarg@cea.fr
jmombarg.github.io/PersonalWebsite/

22/07/25

Funded by the European Union

Joey Mombarg


FACT:
STARS ROTATE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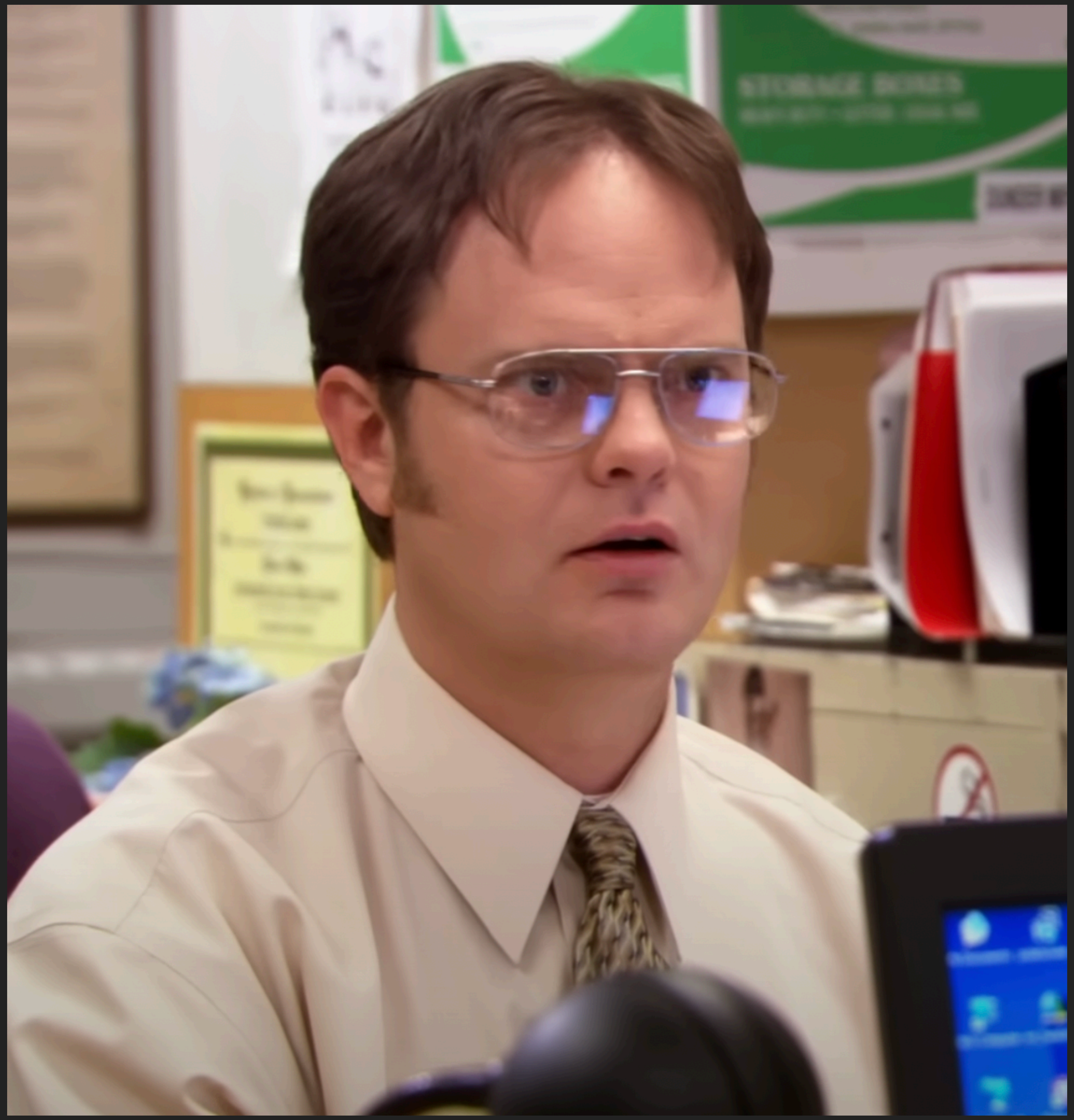

GOALS FOR TODAY

- Learn how rotation and angular momentum transport works in MESA.
- Compare 1D (MESA) with 2D (ESTER) predictions.

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- Learn how rotation and angular momentum transport works in MESA.
- Compare 1D (MESA) with 2D (ESTER) predictions.

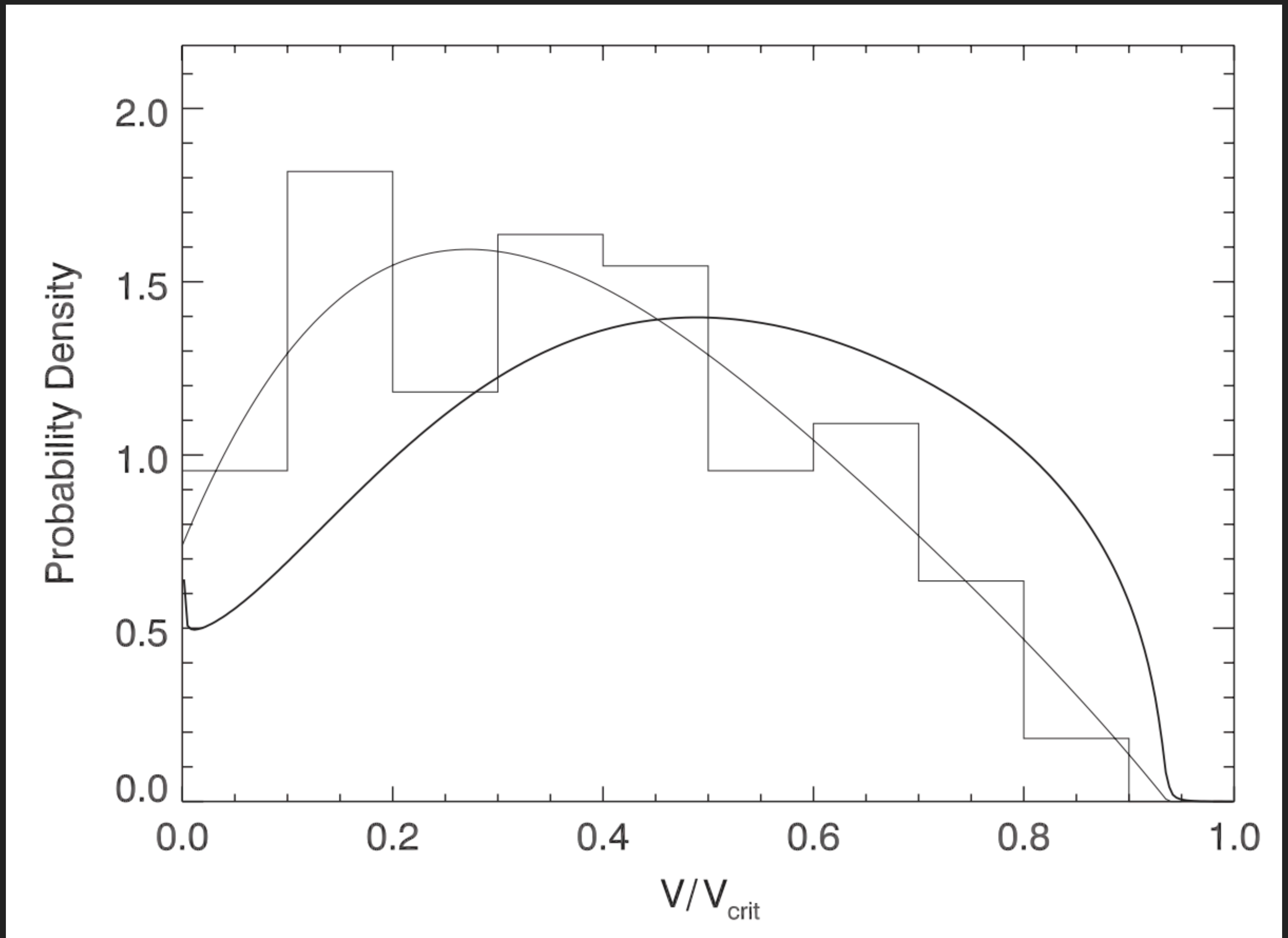
Lab 1: Run a rotating model with angular momentum transport.
Lead TA: Beatriz Bordadágua

Lab 2: Compute the advection velocity and compare with 2D models.
Lead TA: Philip Mocz

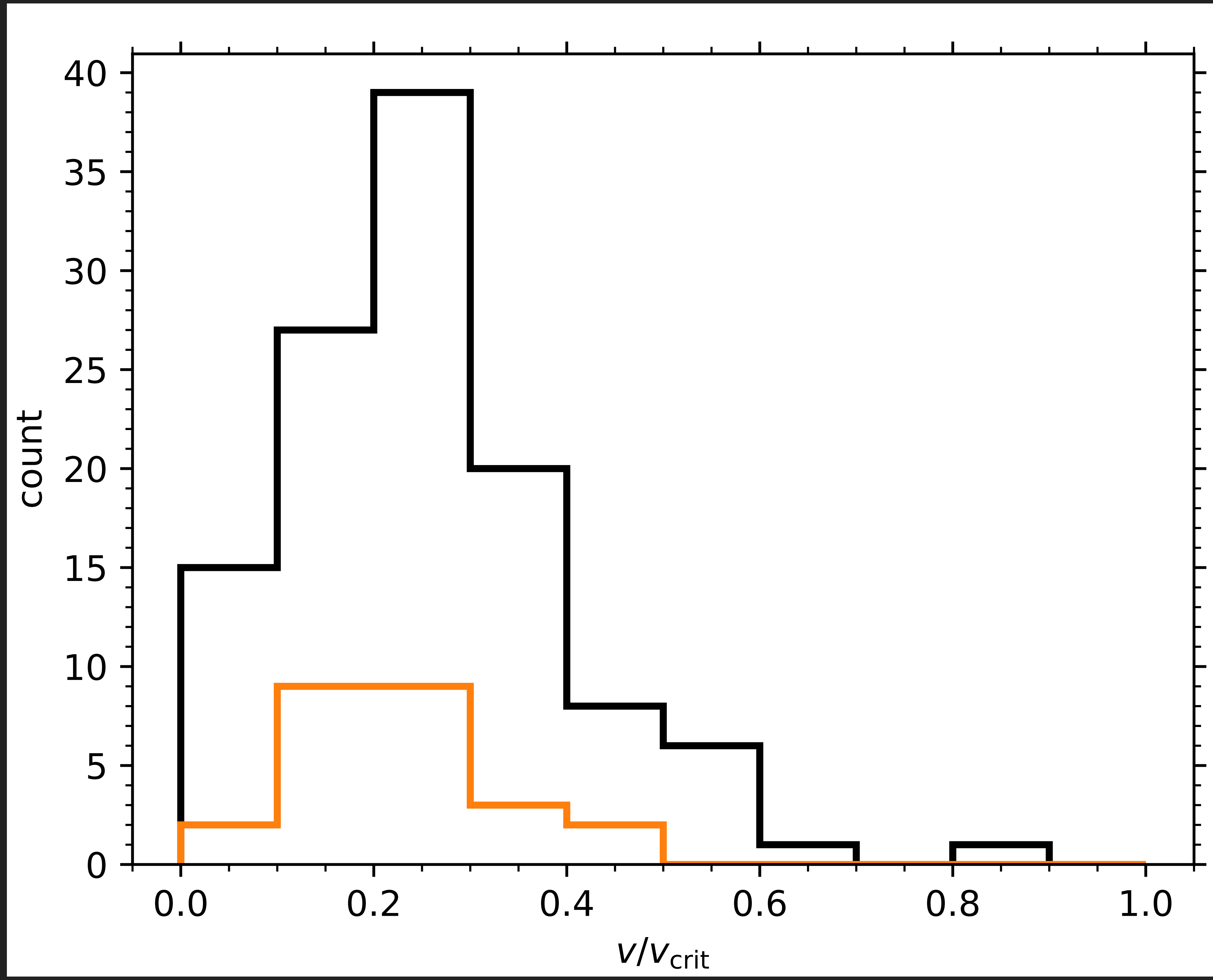
Lab 3: Implement advection (extra torque) in `run_star_extras`.
Lead TA: Tryston Raecke

HOW FAST DO STARS ROTATE?

Young B-type stars



Huang et al. (2010)



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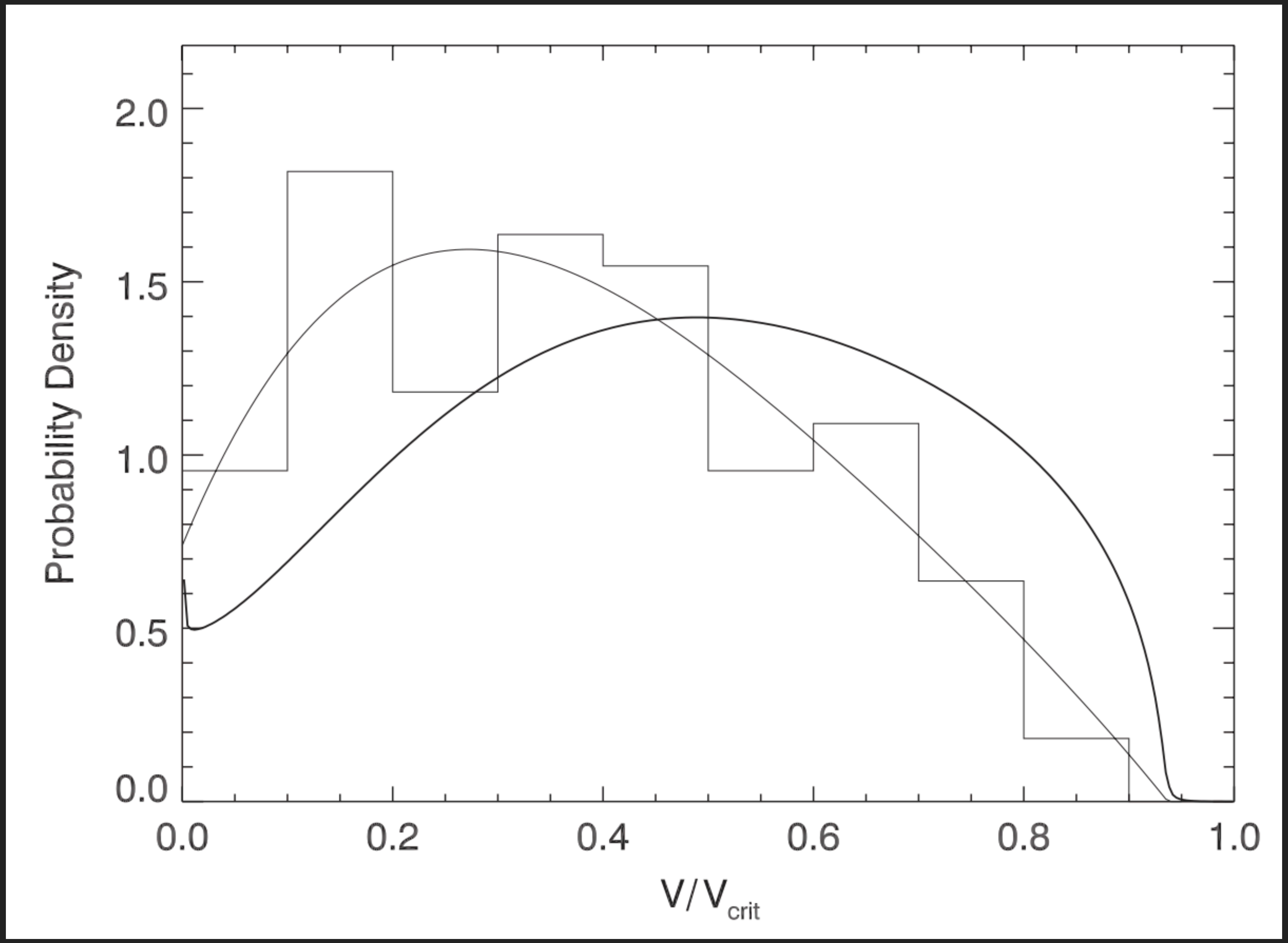
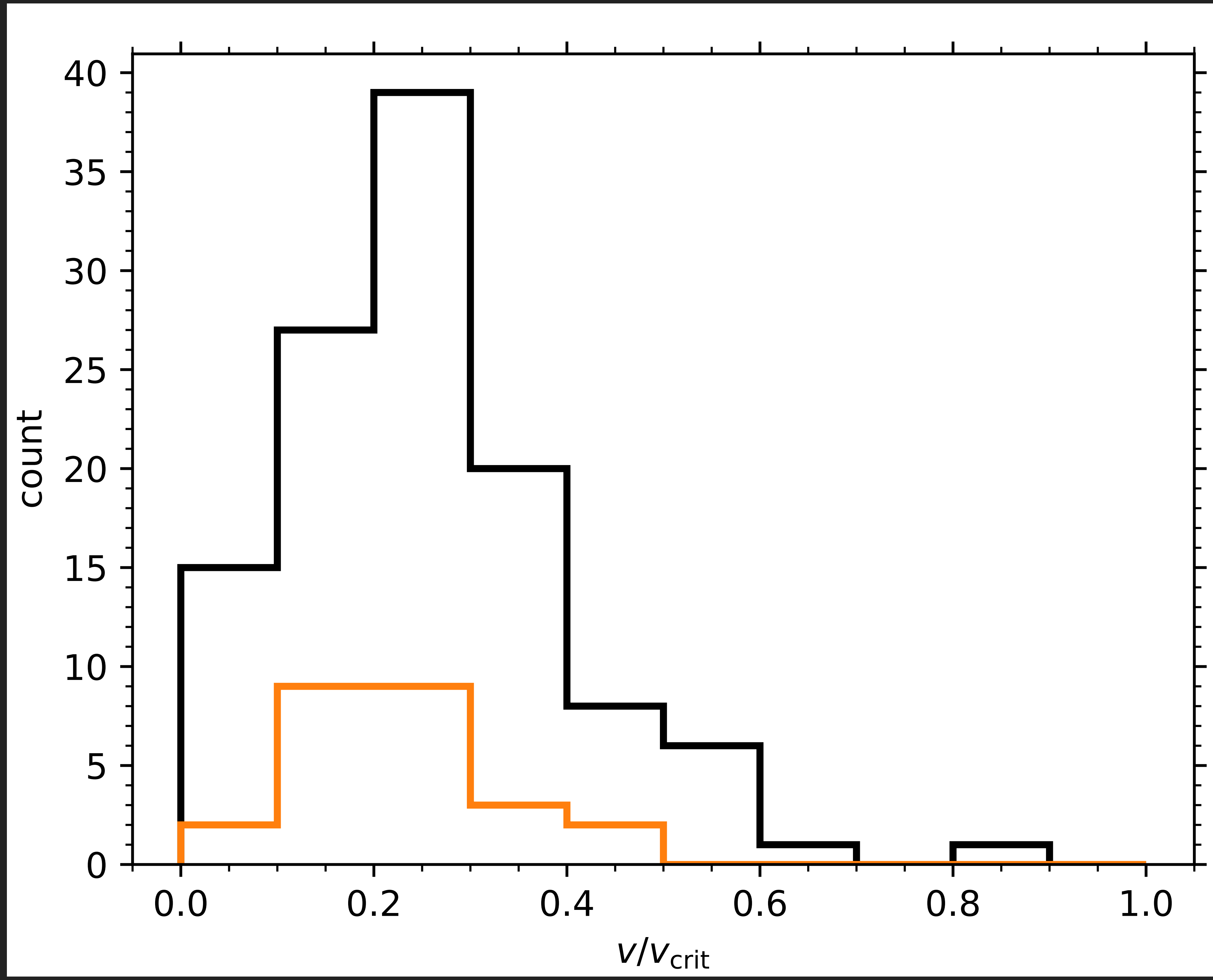


Huang et al. (2010)

β Cephei (OB) pulsators, current



Fritzewski et al. (2025)

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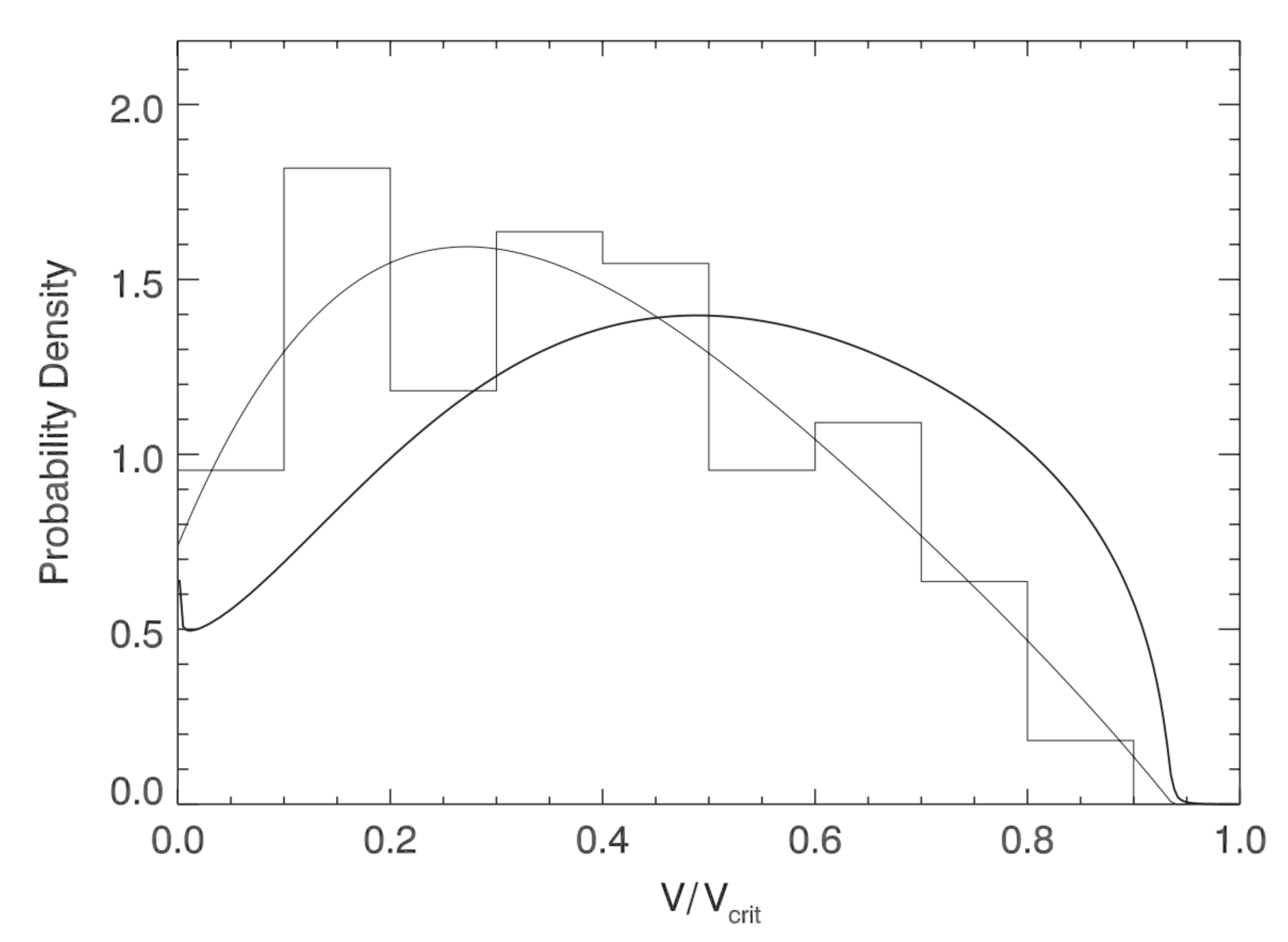

Huang et al. (2010)

$$V_{\rm crit}^{\rm Roche} = \sqrt{\frac{2GM}{3R_{\rm pole}}}$$

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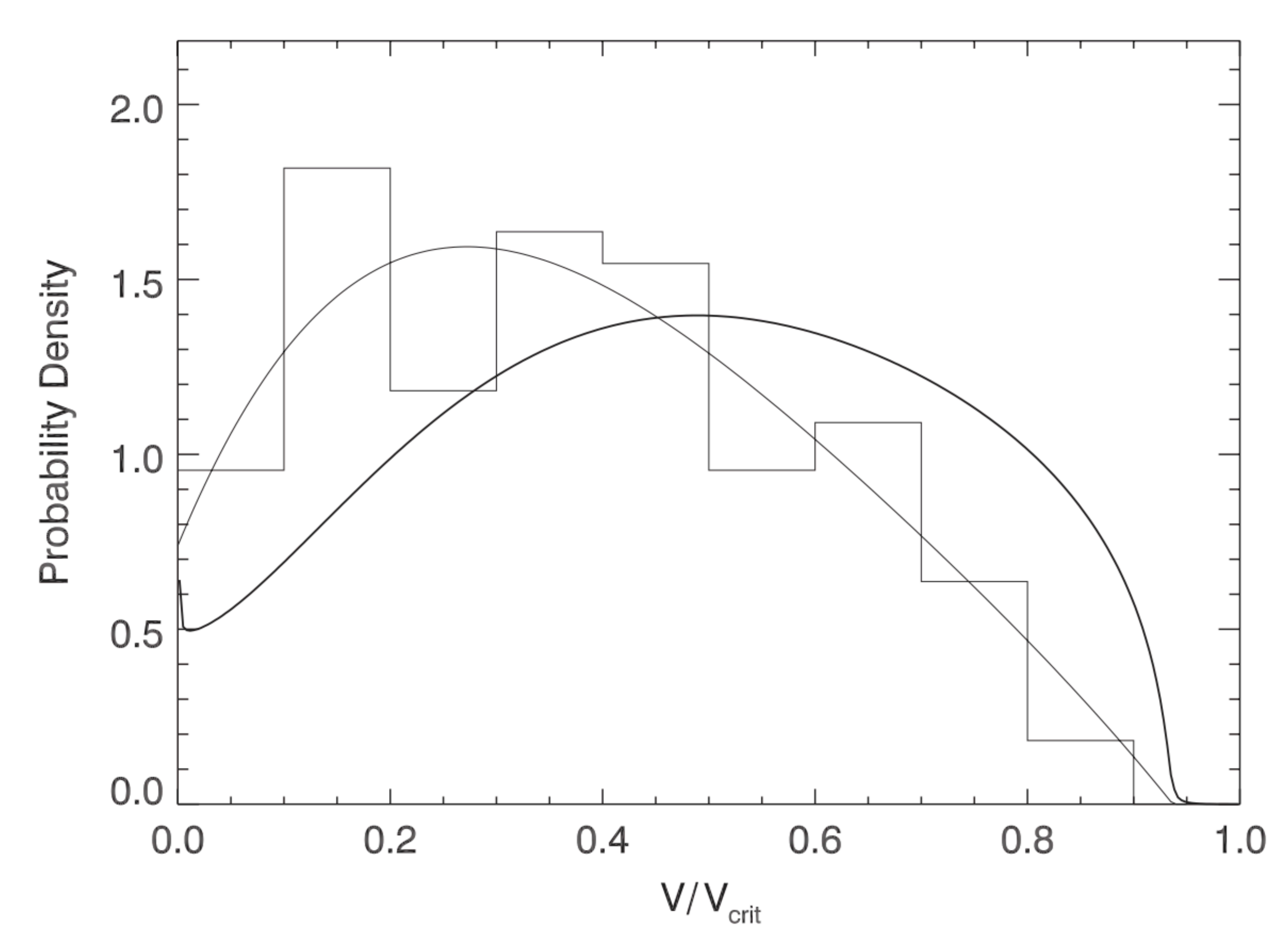
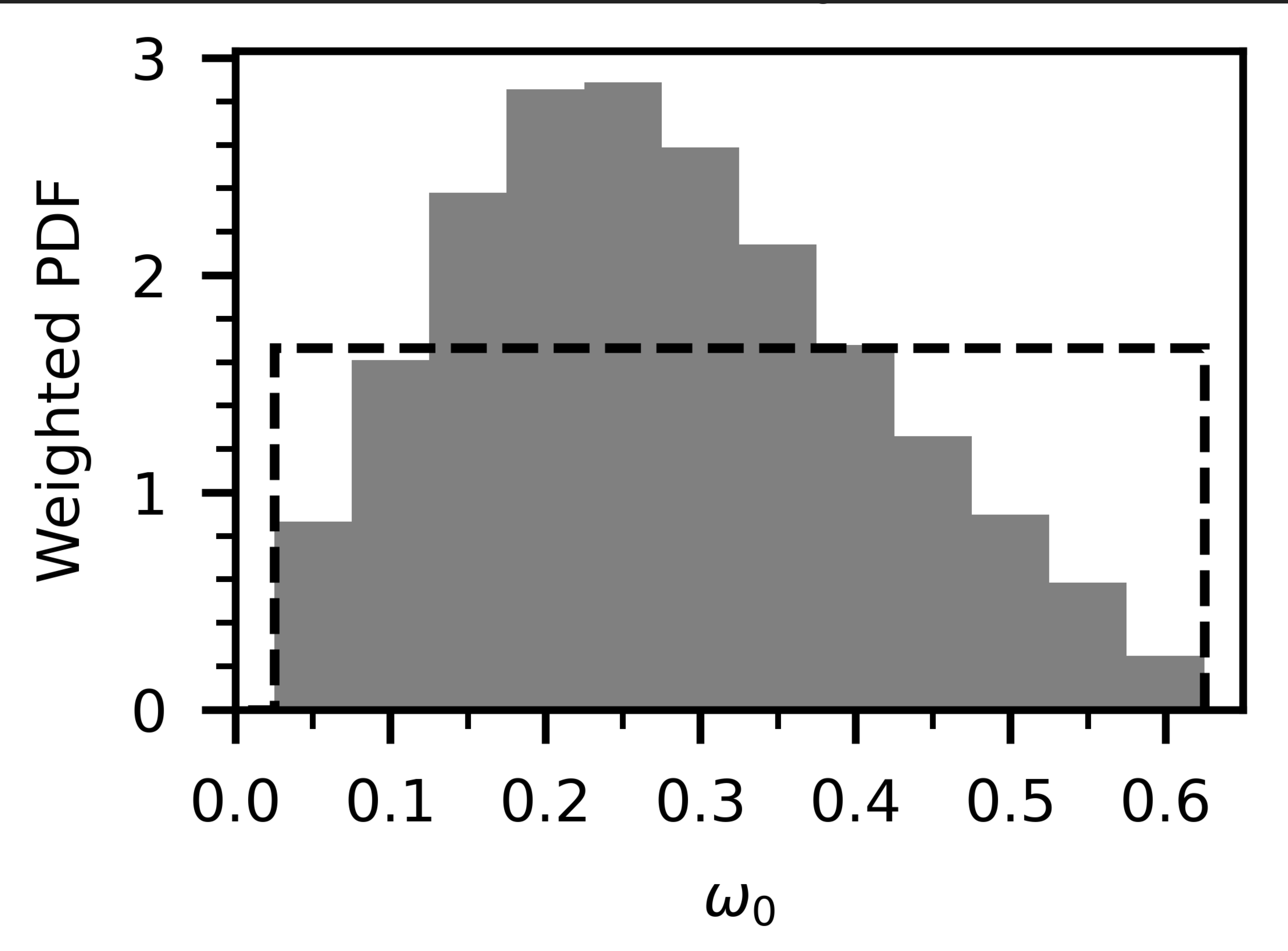


Huang et al. (2010)

F-type stars at ZAMS, backtracked



Mombarg et al. (2024a)

$$V_{\rm crit}^{\rm Roche} = \sqrt{\frac{2GM}{3R_{\rm pole}}}$$

$$\Omega_{\rm crit}^{\rm Kepler} = \sqrt{\frac{GM}{R_{\rm eq}^3}}$$

CRITICAL ROTATION FREQUENCY

- Frequency at which the centrifugal acceleration equals the gravitational one at the equator.

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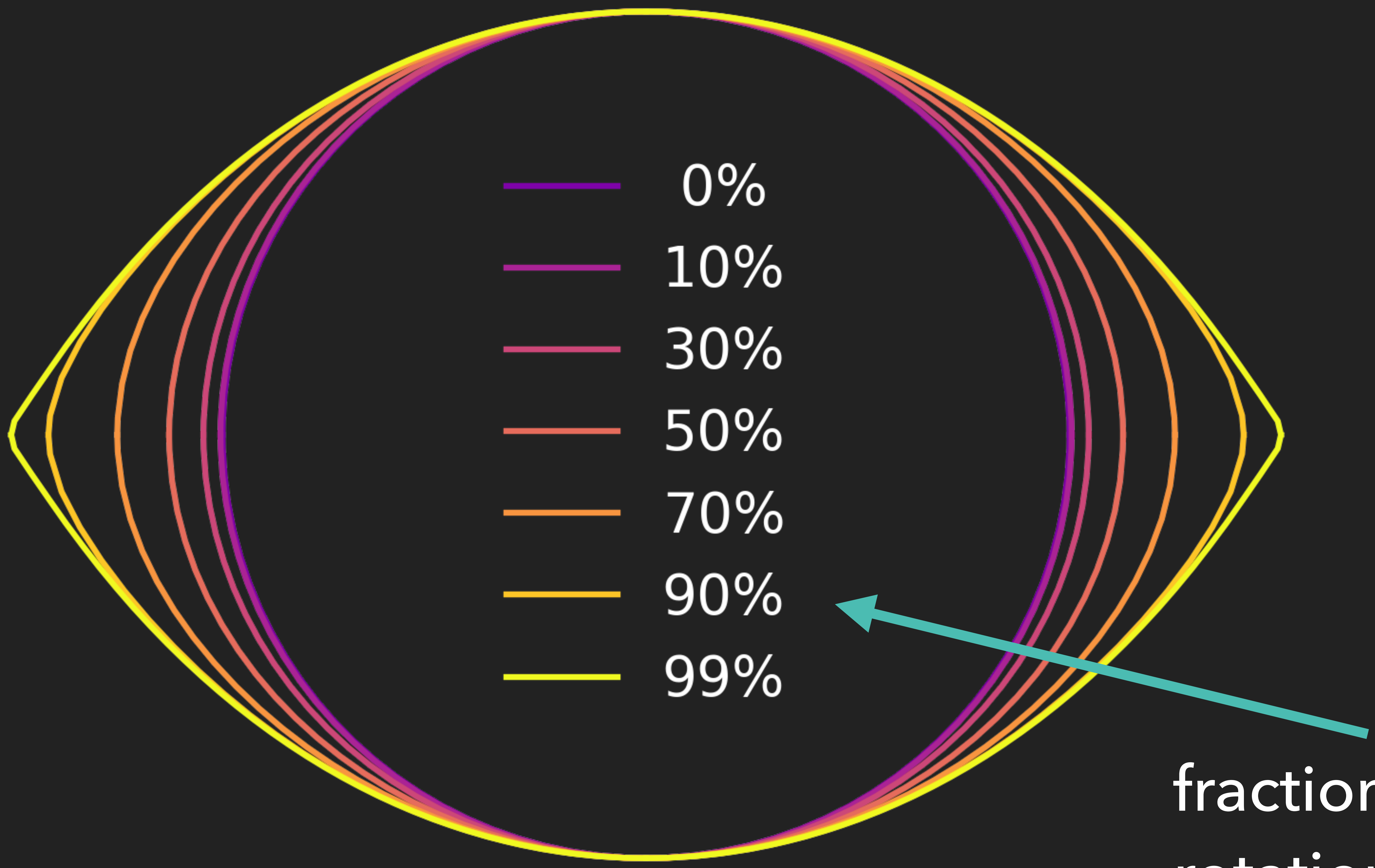
$$\Omega_{\rm crit}^{\rm Kepler} = \sqrt{\frac{GM}{R_{\rm eq}^3}}$$

$$\Omega_{\rm crit}^{\rm Roche} = \sqrt{\frac{8GM}{27R_{\rm pole}^3}}$$

$$\Omega_{\rm crit}^{\rm MESA} = \sqrt{\frac{GM}{R_{\rm eq}^3}\left(1 - \frac{L_\star}{L_{\rm Edd}}\right)}$$

ROTATIONAL DEFORMATION

- Centrifugal force flattens the spherical shape in rotating stars.



fraction of the break-up rotation frequency

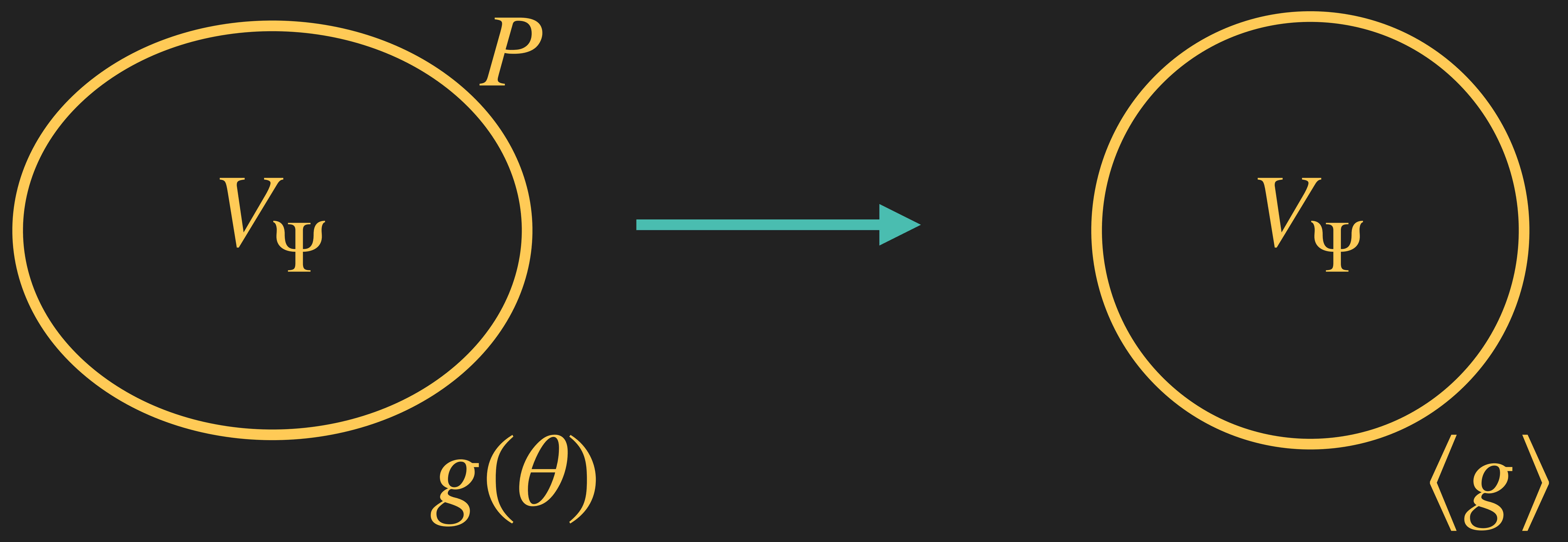
ADJUSTMENTS TO STELLAR STRUCTURE

- Assuming shellular rotation.

$$r \rightarrow r_\Psi, \quad V_\Psi = \frac{4}{3}\pi r_\Psi^3$$

mass conservation

$$\frac{\partial m_\Psi}{\partial r_\Psi} = 4\pi r_\Psi^2 \rho$$



Based on Endal & Sofia (1976)

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$$\frac{\partial m_\Psi}{\partial r_\Psi} = 4\pi r_\Psi^2 \rho$$

hydrostatic equilibrium

$$\frac{\partial P}{\partial m_\Psi} = \frac{G m_\Psi}{4\pi r_\Psi^4} f_P - \frac{1}{4\pi r_\Psi^2}\left(\frac{\partial^2 r_\Psi}{\partial t^2}\right)_{m_\Psi}$$

energy transport

$$\frac{\partial \ln T}{\partial \ln P} = \frac{3\kappa P L_\Psi}{16\pi a c G T^4 m_\Psi} \frac{f_T}{f_P}\left(1 + \frac{r_\Psi^2}{G m_\Psi f_P}\left(\frac{\partial^2 r_\Psi}{\partial t^2}\right)_{m_\Psi}\right)^{-1}$$

Based on Endal & Sofia (1976)

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$$f_P \equiv \frac{4\pi r_\Psi^4}{G m_\Psi S_\Psi} \left\langle g^{-1} \right\rangle^{-1}$$

$$f_T \equiv \left(\frac{4\pi r_\Psi^2}{S_\Psi}\right)^2 \left(\langle g \rangle \left\langle g^{-1} \right\rangle\right)^{-1}$$

Based on Endal & Sofia (1976)

ADJUSTMENTS TO STELLAR STRUCTURE

$$f_P \equiv \frac{4\pi r_\Psi^4}{G m_\Psi S_\Psi} \left\langle g^{-1} \right\rangle^{-1} \qquad f_T \equiv \left(\frac{4\pi r_\Psi^2}{S_\Psi} \right)^2 \left(\langle g \rangle \left\langle g^{-1} \right\rangle \right)^{-1}$$

- In MESA, $f_P(\omega)$ and $f_T(\omega)$ are computed from polynomials prefitted to Roche models. See Appendix B.4 of Paxton et al. (2019, Paper V).

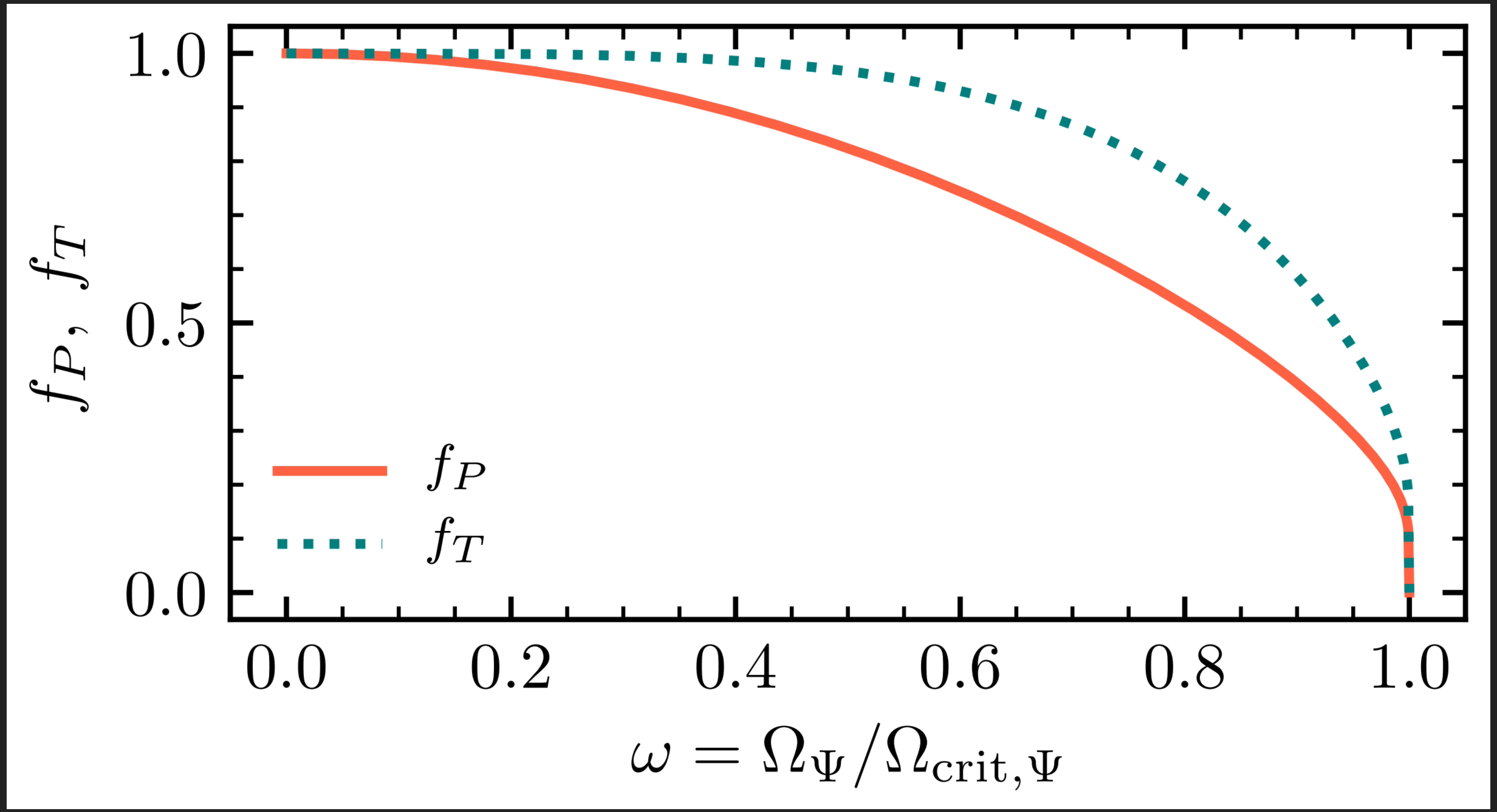
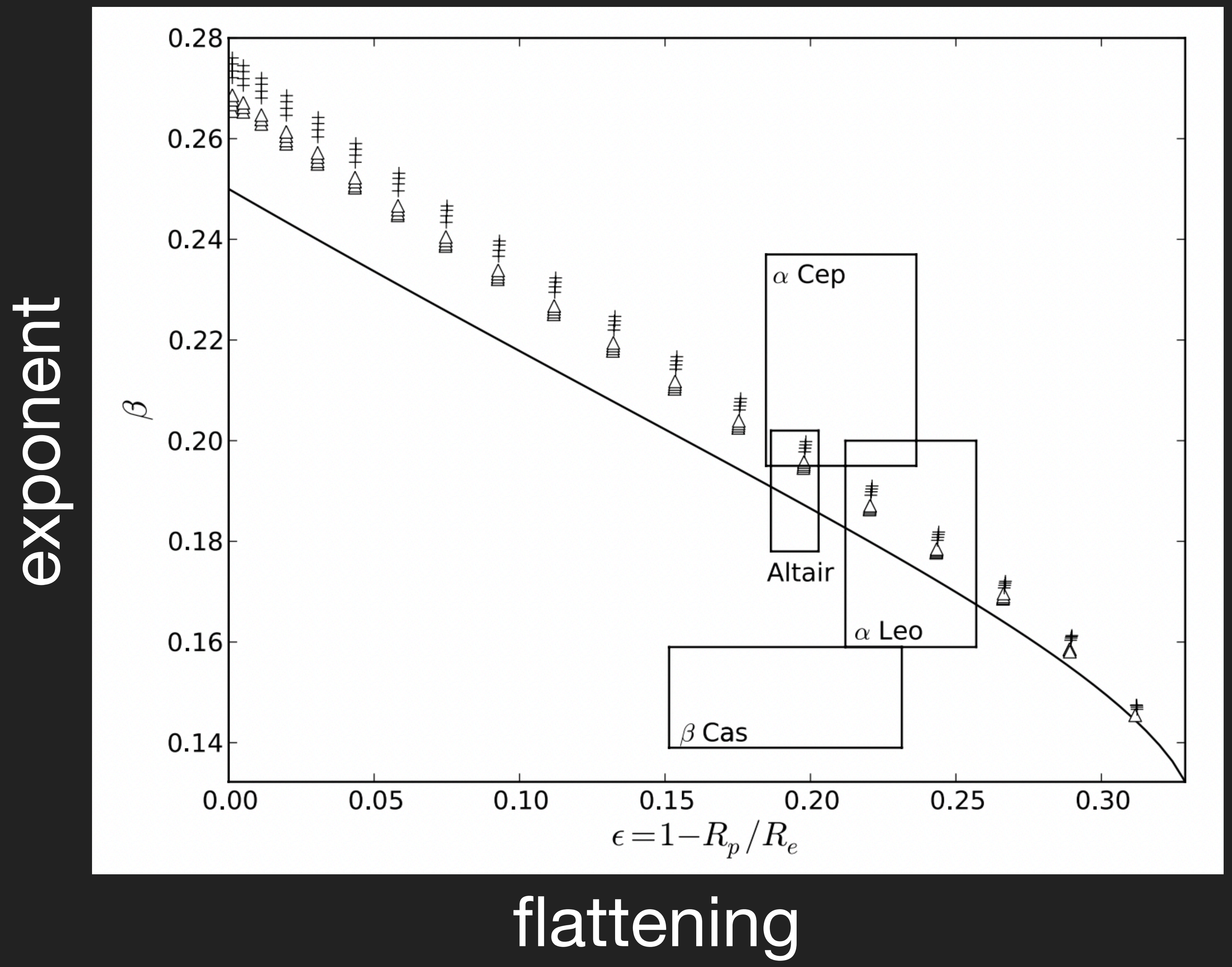


Fig. 33
MESA Paper V

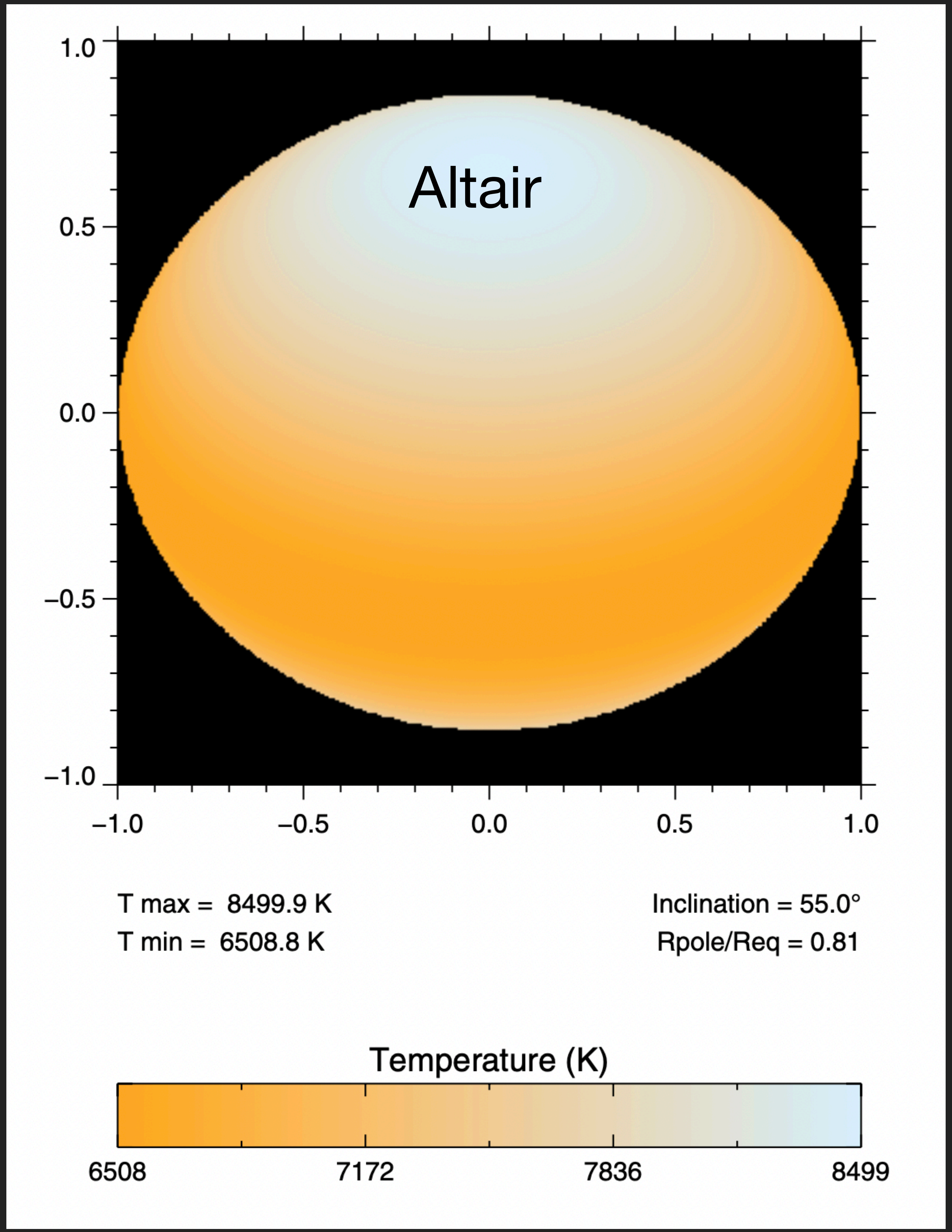
GRAVITY DARKENING

- Von Zeipel (1924) theorem

$$T_{\rm eff} \propto g_{\rm eff}^{1/4}$$



Espinosa Lara & Rieutord (2011)



Domiciano de Souza et al. (2005)

- Exponent depends on the flattening.

PROJECTION EFFECTS

- Gravity darkening coefficients are tabulated in MESA.

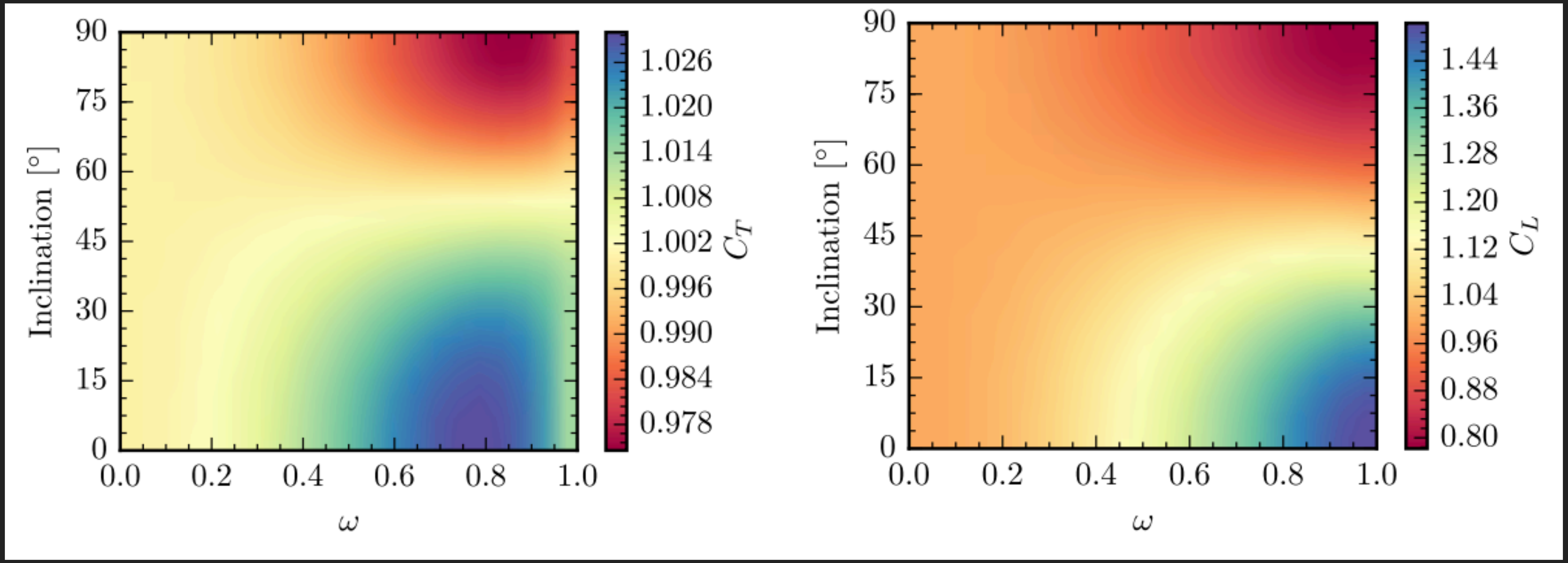


Fig. 37, MESA Paper V

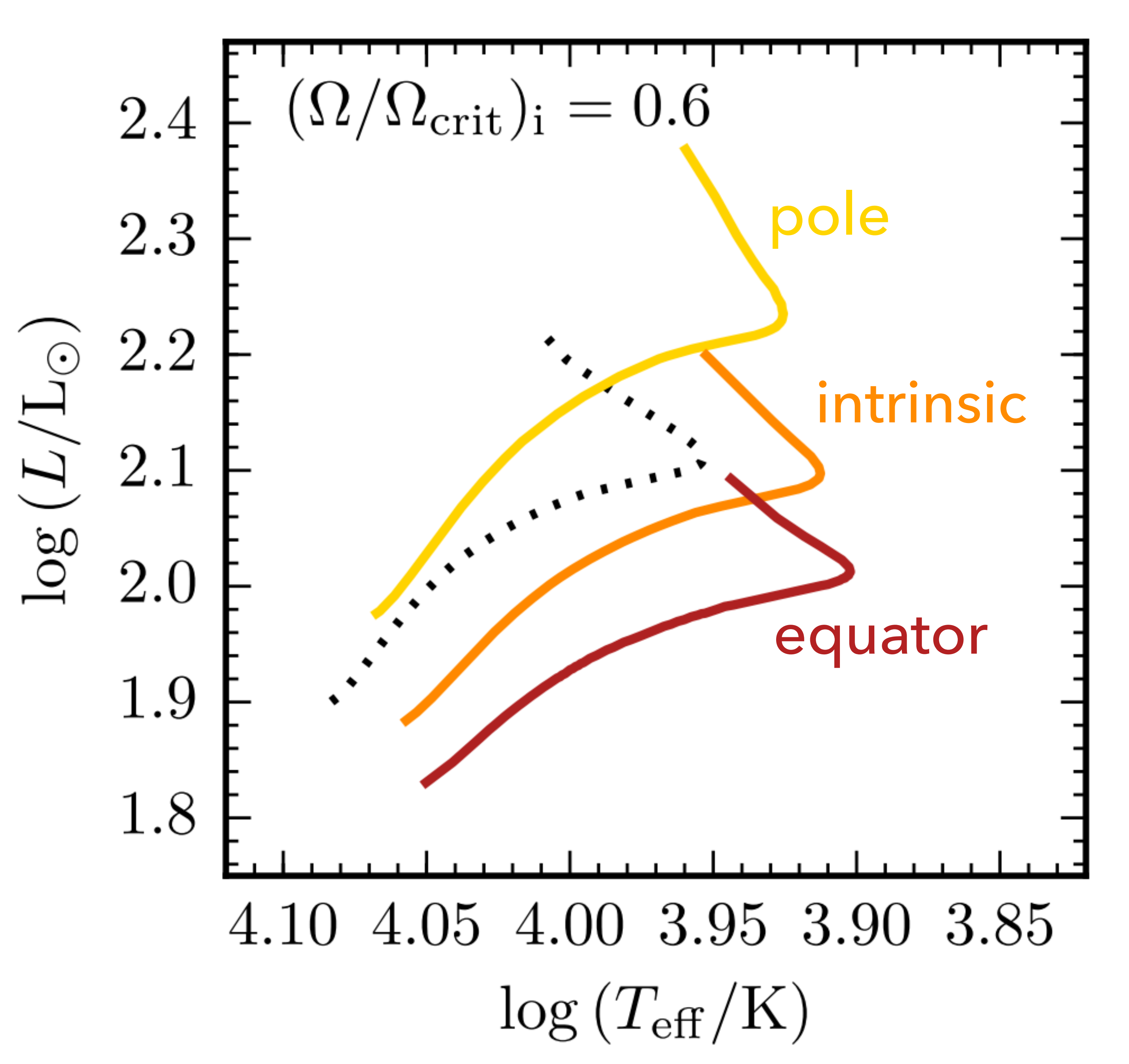


Fig. 38, MESA Paper V

```
gravity_dark_L_polar
gravity_dark_Teff_polar
gravity_dark_L_equatorial
gravity_dark_Teff_equatorial
```

history_columns.list

ANGULAR MOMENTUM (AM) TRANSPORT

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\Omega\right) = \underbrace{\frac{1}{r^2}\partial_r\left(\rho\nu r^4\partial_r\Omega\right)}_{\text{diffusion}} + \underbrace{\frac{1}{5r^2}\partial_r\left(\rho r^4\Omega U\right)}_{\text{advection}} - \underbrace{\frac{1}{r^2}\partial_r\left(r^2F_J(r)\right)}_{\text{internal gravity waves}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \dot{r}\partial_r$$

ANGULAR MOMENTUM (AM) TRANSPORT IN MESA

- MESA AM transport is fully diffusive.

$$\left(\frac{\partial \Omega}{\partial t}\right)_m = \underbrace{\frac{1}{i}\left(\frac{\partial}{\partial m}\right)_t \left[(4\pi r^2 \rho)^2 i \nu_{\rm AM} \left(\frac{\partial \Omega}{\partial m}\right)_t\right]}_{\text{AM diffusion}} - \underbrace{\frac{\Omega}{r}\left(\frac{\partial r}{\partial t}\right)_m \left(\frac{\mathrm{d}\ln i}{\mathrm{d}\ln r}\right)}_{\text{expansion/contraction}}$$

specific moment of inertia (i)

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$$\left(\frac{\partial \Omega}{\partial t}\right)_m = \frac{1}{i}\left(\frac{\partial}{\partial m}\right)_t \left[(4\pi r^2 \rho)^2 i \nu_{\rm AM} \left(\frac{\partial \Omega}{\partial m}\right)_t\right] - \frac{\Omega}{r}\left(\frac{\partial r}{\partial t}\right)_m \left(\frac{d \ln i}{d \ln r}\right)$$

$\nu_{\rm AM} =$

```
am_nu_non_rotation_factor * am_nu_non_rot +
am_nu_factor * am_nu_rot
```

```
set_uniform_am_nu_non_rot = .true.
uniform_am_nu_non_rot = <>
```

See Heger et al. (2000)

```
am_nu_rot =
am_nu_DSI_factor * D_DSI +
am_nu_SH_factor  * D_SH  +
am_nu_SSI_factor * D_SSI +
am_nu_ES_factor  * D_ES  +
am_nu_GSF_factor * D_GSF +
am_nu_ST_factor  * D_ST
```

ENABLE ROTATION IN MESA

- Rotation starts at ZAMS and is uniform.

```
&controls

L_nuc_div_L_zams_limit = 0.9d0
```

1. Equatorial velocity (km/s).

```
&star_job

new_surface_rotation_v
```

ENABLE ROTATION IN MESA

- Rotation starts at ZAMS and is uniform.

```
&controls

L_nuc_div_L_zams_limit = 0.9d0
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1. Equatorial velocity (km/s).
2. Angular frequency (rad/s).

```
&star_job

new_omega
```

ENABLE ROTATION IN MESA

- Rotation starts at ZAMS and is uniform.

```
&controls

L_nuc_div_L_zams_limit = 0.9d0
```

1. Equatorial velocity (km/s).
2. Angular frequency (rad/s).
3. Fraction of critical rotation frequency (Keplerian).

```
&star_job

new_omega_div_omega_crit
```

```
set_near_zams_omega_div_omega_crit_steps = 15
near_zams_relax_omega_div_omega_crit = .true.
```

BAROTROPIC AND BAROCLINIC STARS

- Start from equation of hydrostatic equilibrium

$$\frac{1}{\rho}\nabla P = -\nabla\Phi + s\Omega^2\hat{s} \qquad s = r\sin\theta$$

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curl

$$-\frac{1}{\rho^2}\nabla\rho\times\nabla P = \underbrace{-\nabla(s\Omega^2)\times\nabla s}_{\text{zero if } \Omega = \Omega(s)}$$

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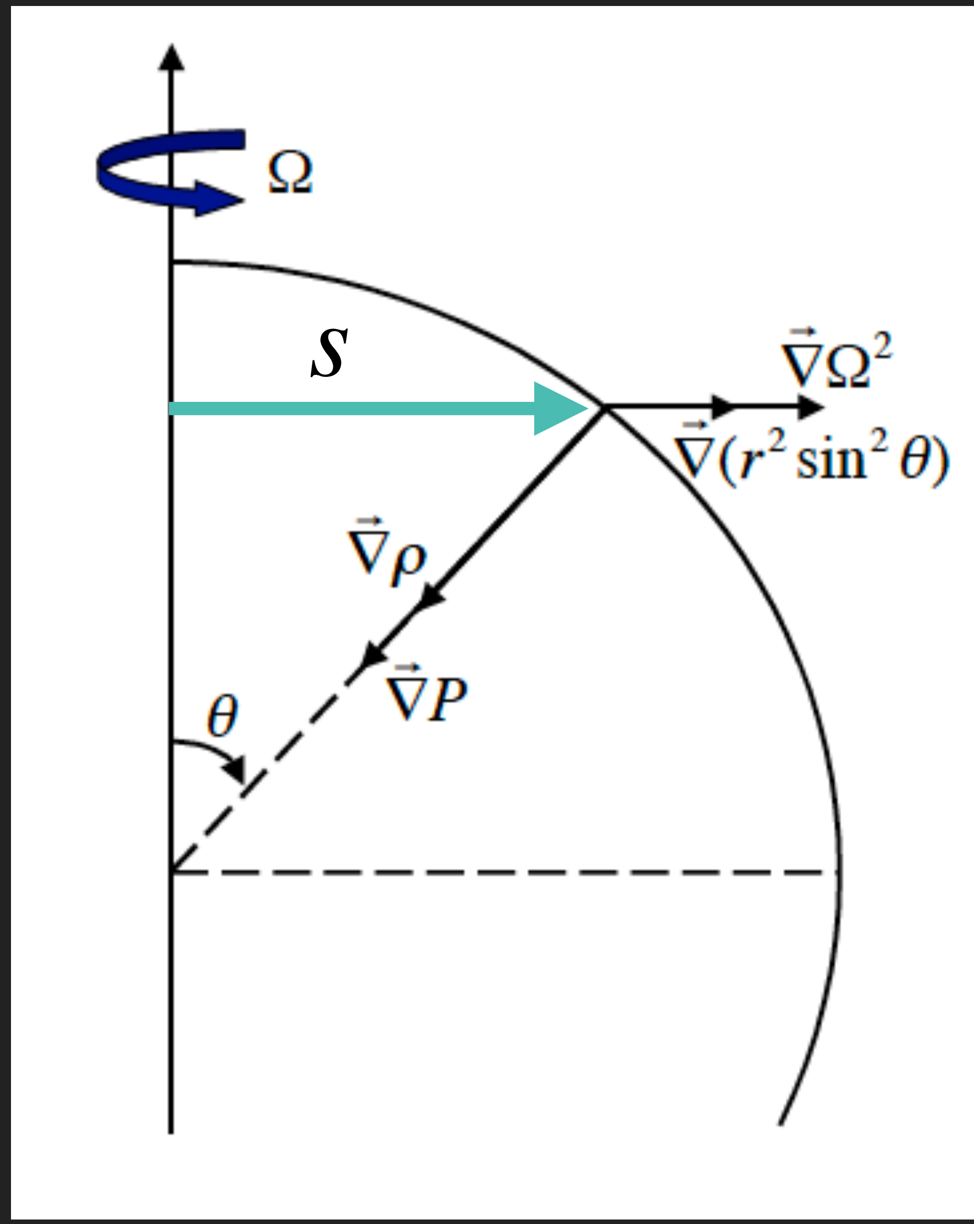

PhD thesis S. Mathis (2005)

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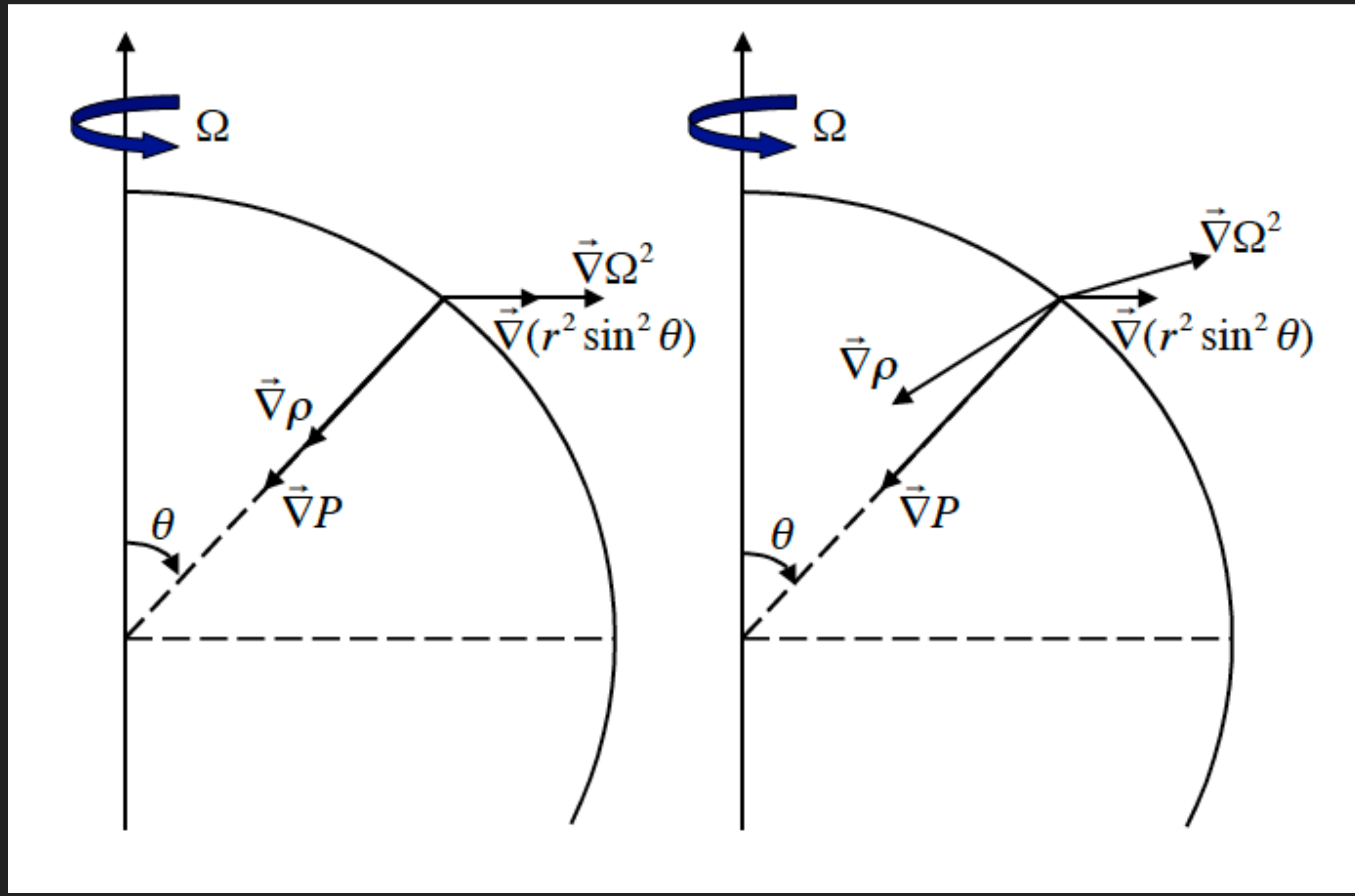

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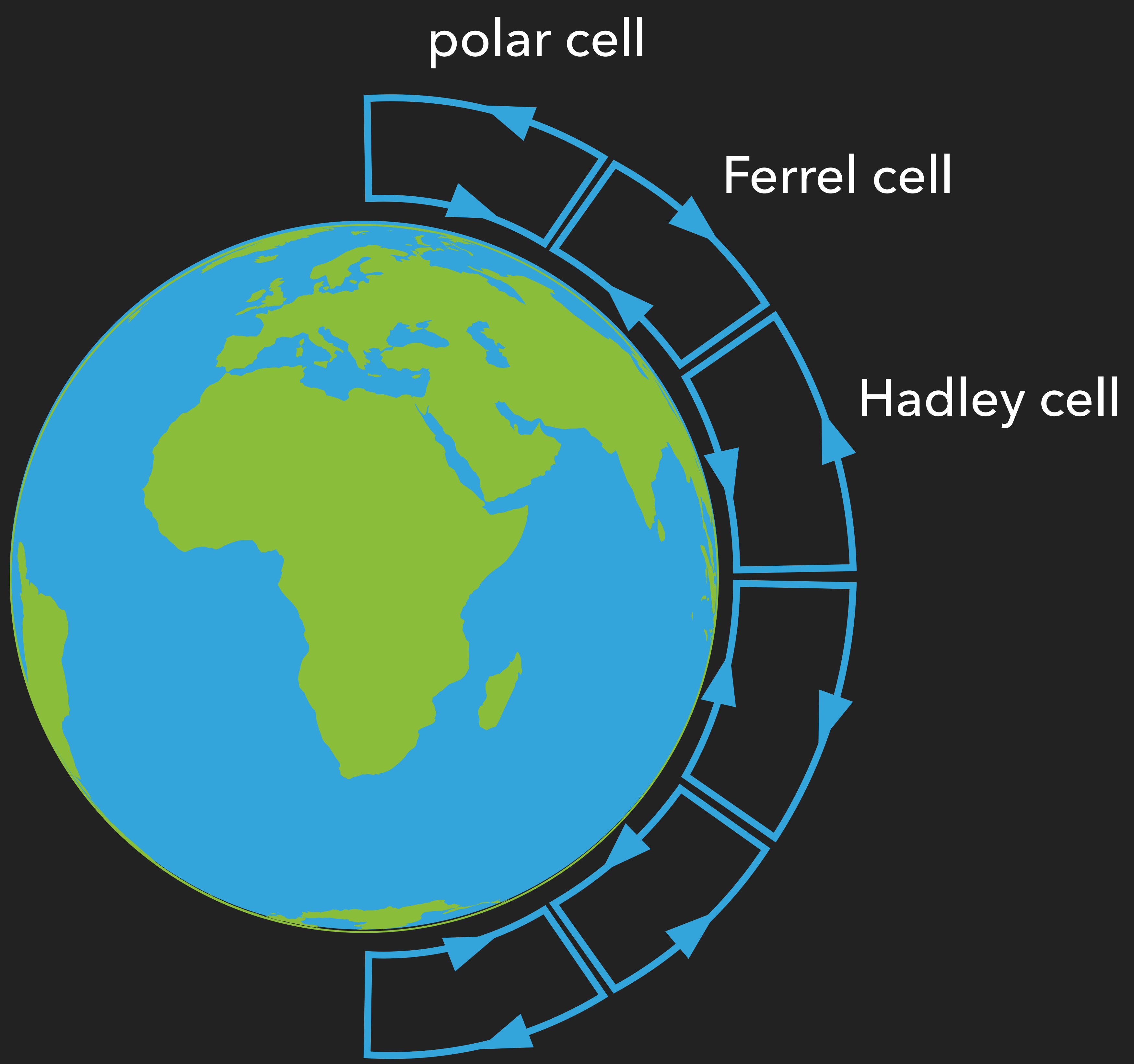
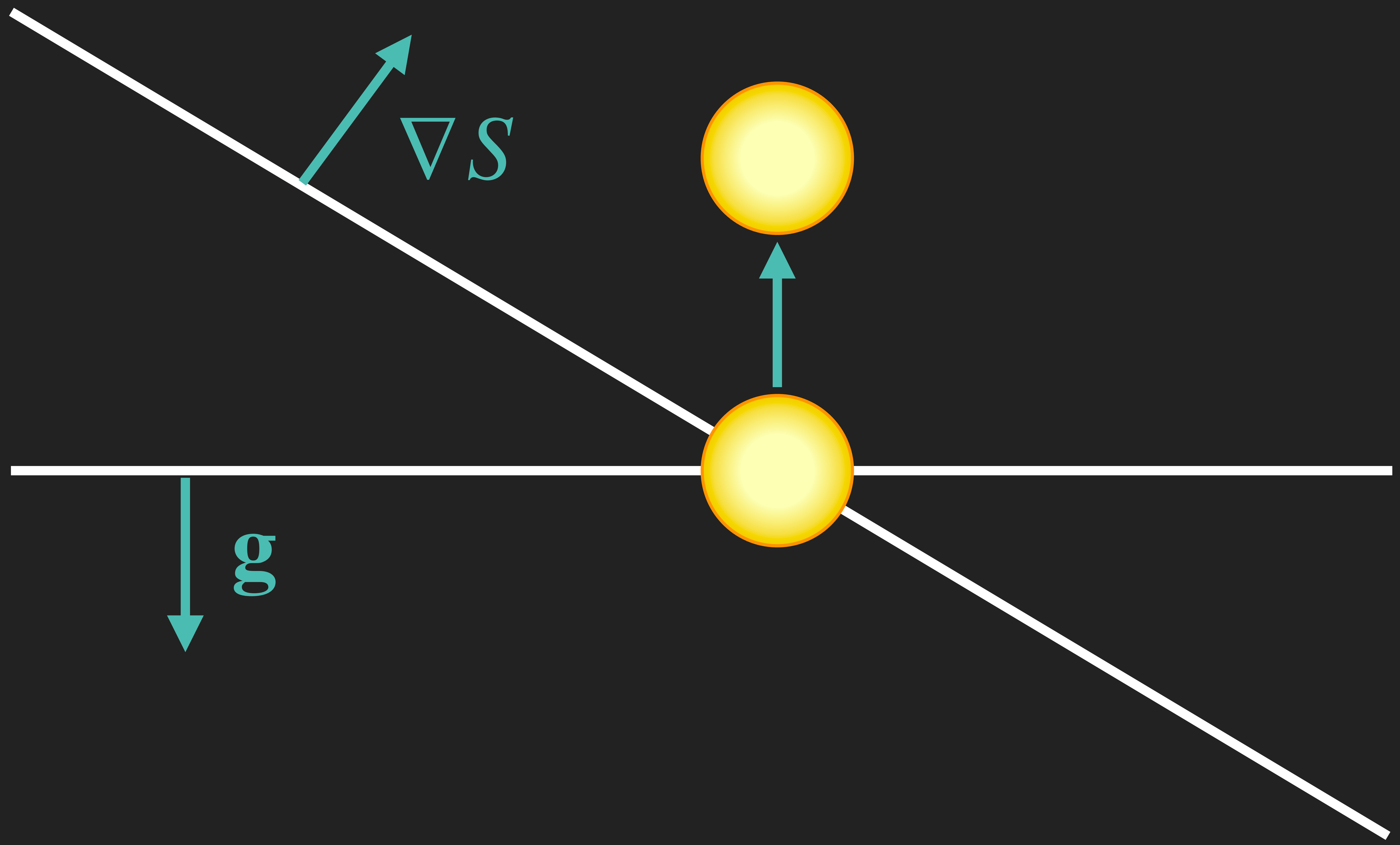
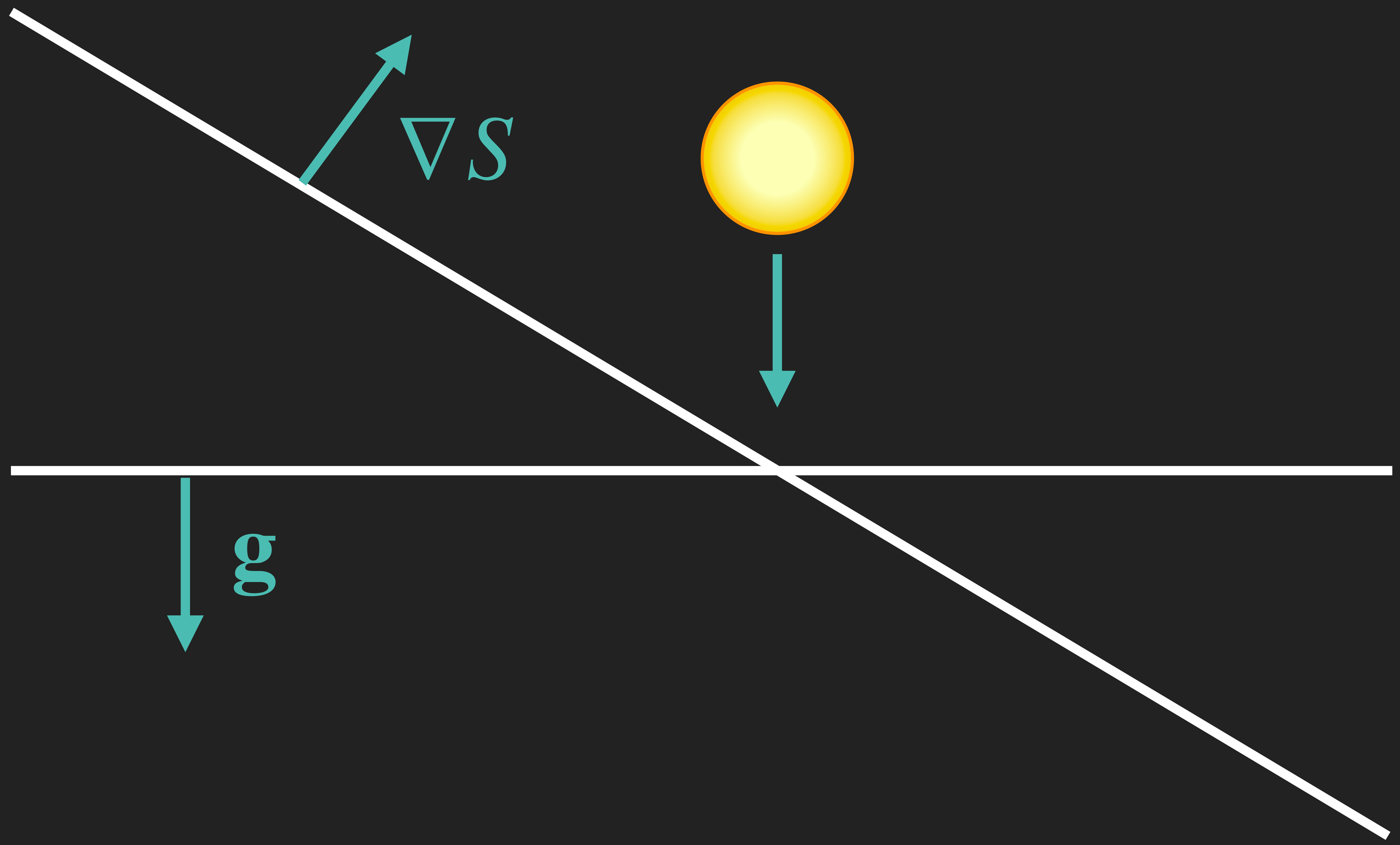


- Baroclinicity drives the jet stream on Earth.

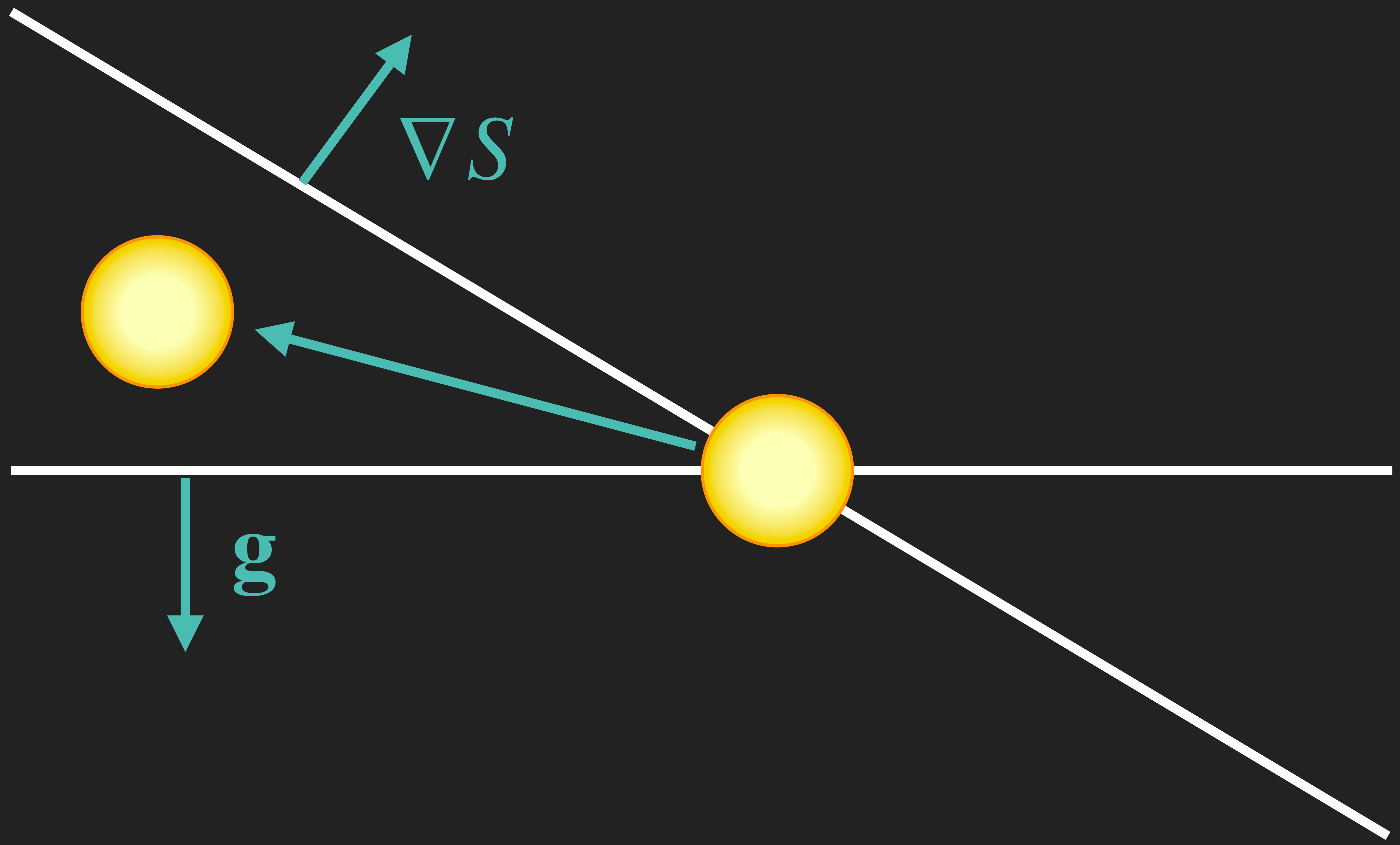
BAROCLINIC INSTABILITY



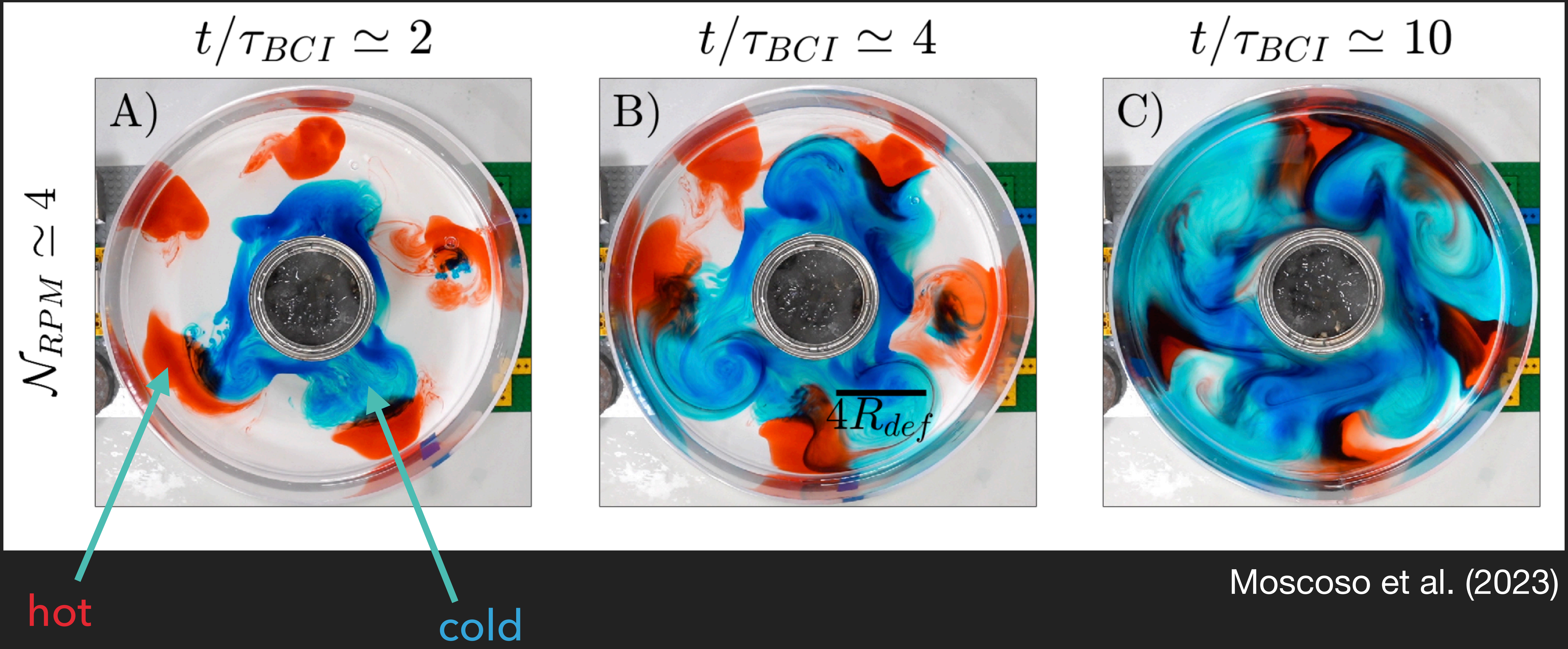
BAROCLINIC INSTABILITY



BAROCLINIC INSTABILITY



BAROCLINIC INSTABILITY



Moscoso et al. (2023)

GOLDREICH-SCHUBERT-FRICKE (GSF) INSTABILITY

Goldreich & Schubert (1967); Fricke et al. (1968)

- Normally, axisymmetric instabilities are stable. If AM decreases outwards, axisymmetric perturbations can become unstable.

$$\frac{\nu}{K} N_T^2 + N_\Omega^2 < 0 \quad \text{or} \quad \left| s\partial_z \Omega^2 \right| > \frac{\nu}{K} N_T^2$$

$$N_\Omega^2 = \frac{1}{s^3}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\Omega\right)^2$$

$$N_T^2 = \frac{g}{H_P}[\delta(\nabla_{\mathrm{ad}} - \nabla)]$$

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$$N_\Omega^2 = \frac{1}{s^3} \frac{\mathrm{d}}{\mathrm{d}s} \left(s^2 \Omega \right)^2$$

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Goldreich & Schubert (1967);
Fricke et al. (1968)

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$$N_\Omega^2 < 0 \quad \text{or} \quad \left| s\partial_z \Omega^2 \right| > 0$$

$$N_\Omega^2 = \frac{1}{s^3}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\Omega\right)^2$$

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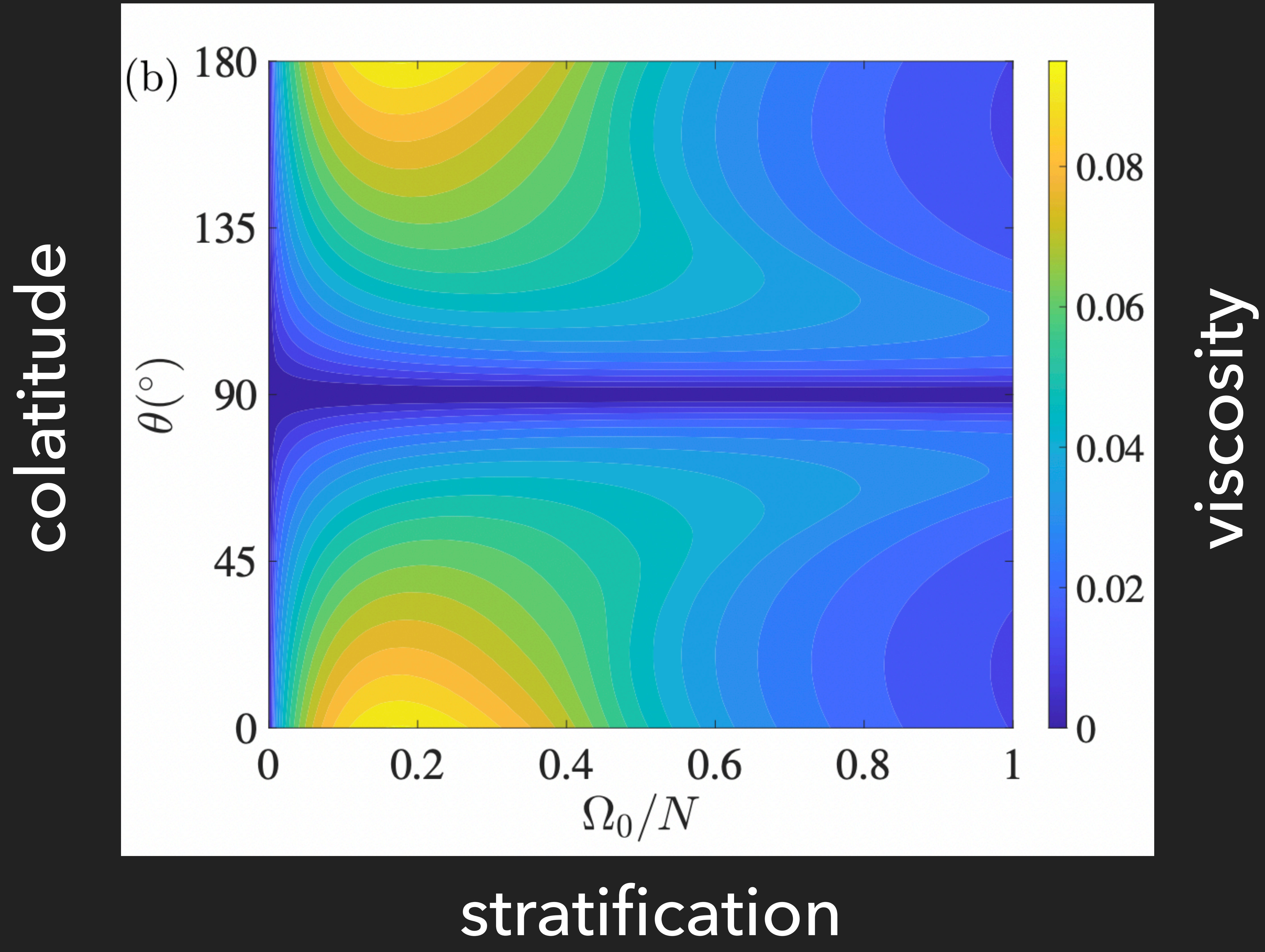
Goldreich & Schubert (1967); Fricke et al. (1968)

- Normally, axisymmetric instabilities are stable. If AM decreases outwards, axisymmetric perturbations can become unstable.

$$\frac{\mathrm{d}j}{\mathrm{d}r} < 0$$

- GSF instability most efficient around the poles (if thermal diffusivity is high).

```
am_nu_GSF_factor
D_GSF_factor
```



Park & Mathis (2025)

SOLBERG-HØILAND (SH) INSTABILITY

- Rayleigh instability: centrifugal acceleration acts as restoring force when

$$N_\Omega^2 = \frac{1}{s^3}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\Omega\right)^2 > 0$$

- SH instability: stratification + Rayleigh (Solberg 1936; Høiland 1941)

$$N^2 + N_\Omega^2 > 0$$

- For $\Omega = 0$, we recover the Ledoux criterion.

```
&controls

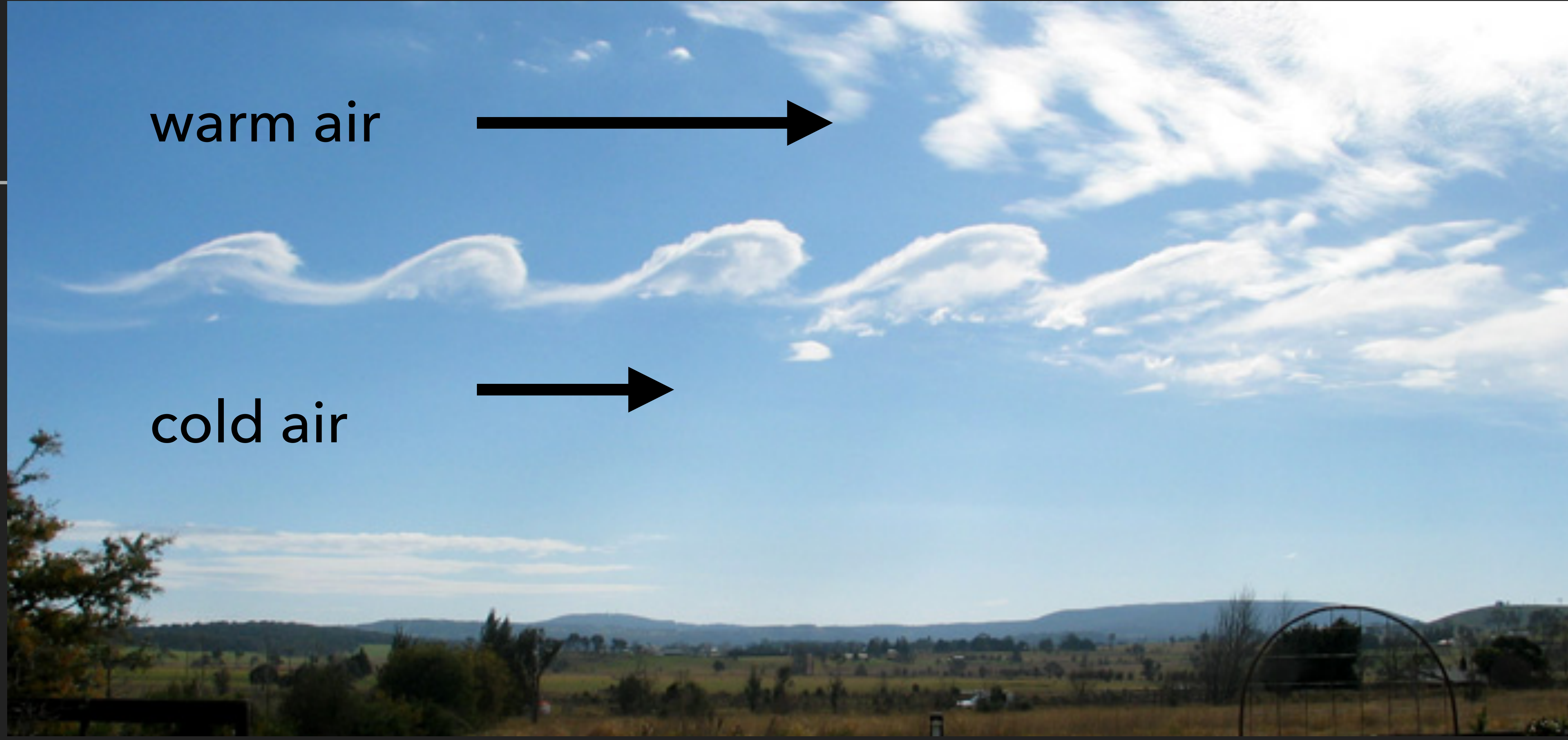
am_nu_SH_factor
D_SH_factor
```

DYNAMICAL SHEAR INSTABILITY (DSI)

- Kelvin-Helmholtz instability

$$\mathrm{Ri} = \frac{N^2}{(\mathrm{d}V/\mathrm{d}z)^2} \geq \mathrm{Ri_c} \leftarrow \frac{1}{4}$$

$\mathrm{d}\Omega/\mathrm{d}\ln r$ in MESA



- Operates on dynamical time scale (fast).

```
&controls

am_nu_DSI_factor
D_DSI_factor
```

SECULAR SHEAR INSTABILITY (SSI)

- Thermal diffusion weakens the stabilising effect of the stratification.
- Two criteria have to be simultaneously violated

Pr: Prandtl number
Re: Reynolds number
Ri: Richardson number

$$\mathrm{Pr} = \frac{\text{viscous diffusivity}}{\text{thermal diffusivity}}, \qquad \mathrm{Re} = \frac{\text{inertial force}}{\text{viscous force}}$$

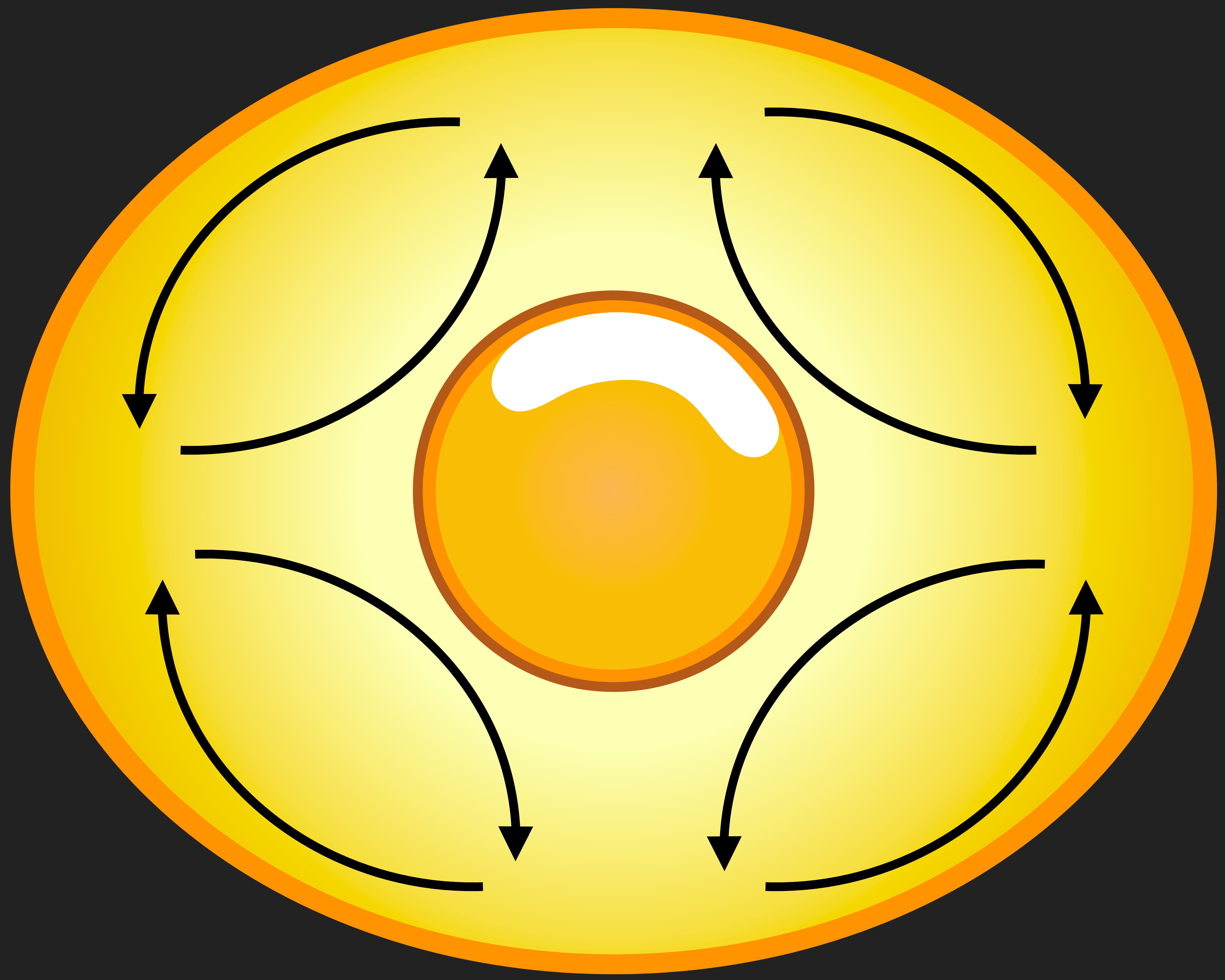
$$\frac{\mathrm{Pr}\,\mathrm{Re_c}}{8} N_T^2 \left(\frac{\mathrm{d}\ln r}{\mathrm{d}\Omega}\right)^2 > \mathrm{Ri_c} \qquad N_\mu^2 \left(\frac{\mathrm{d}\ln r}{\mathrm{d}\Omega}\right)^2 > \mathrm{Ri_c}$$

- SSI operates on thermal time scale. Less efficient than DSI, but requires less strong shear.

```
&controls

am_nu_SSI_factor
D_SSI_factor
```

EDDINGTON-SWEET (ES) CIRCULATION



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&controls

am_nu_ES_factor
D_ES_factor
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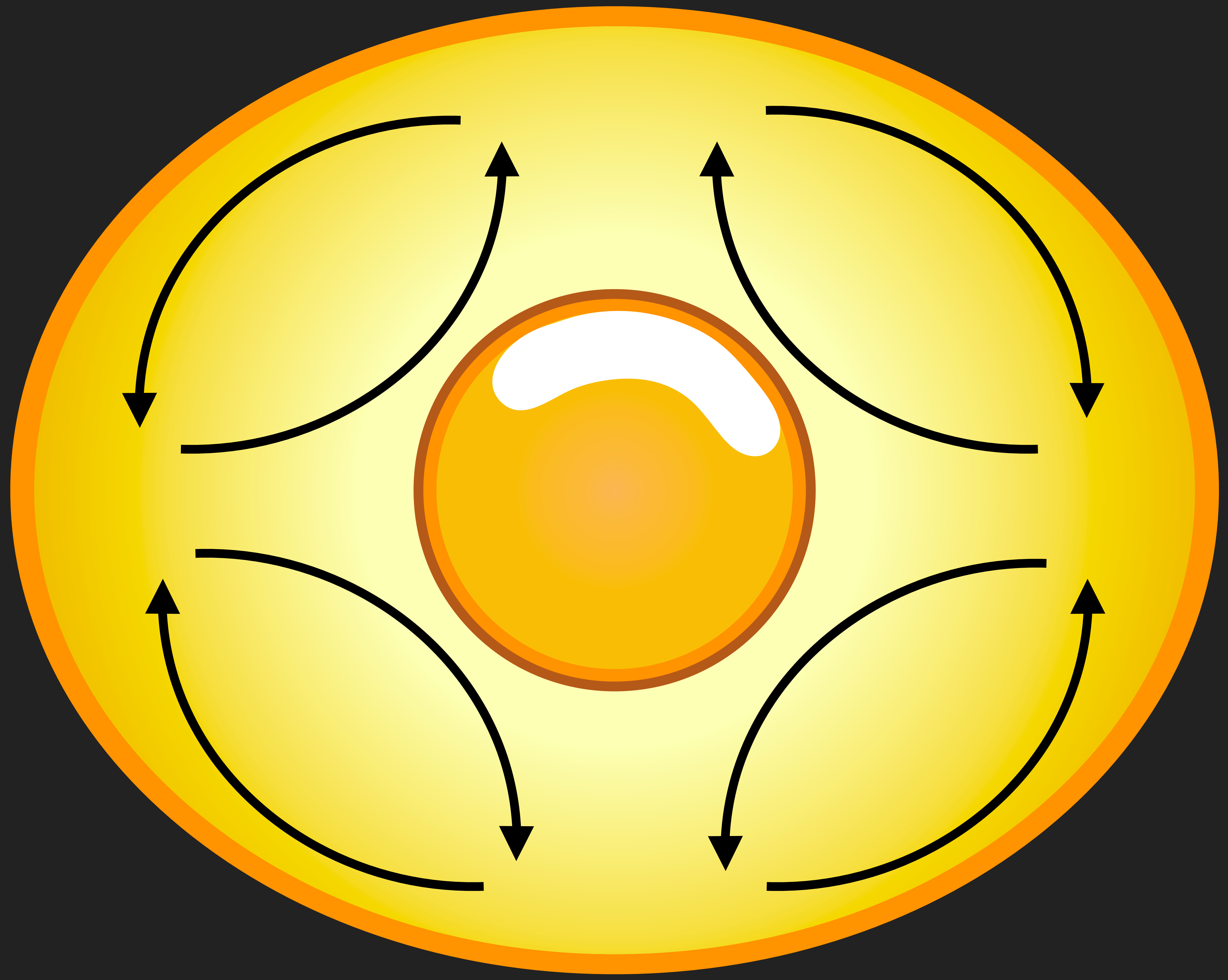
EDDINGTON-SWEET (ES) CIRCULATION

- Rotating star cannot be simultaneously be in hydrostatic and thermal equilibrium.



```
&controls

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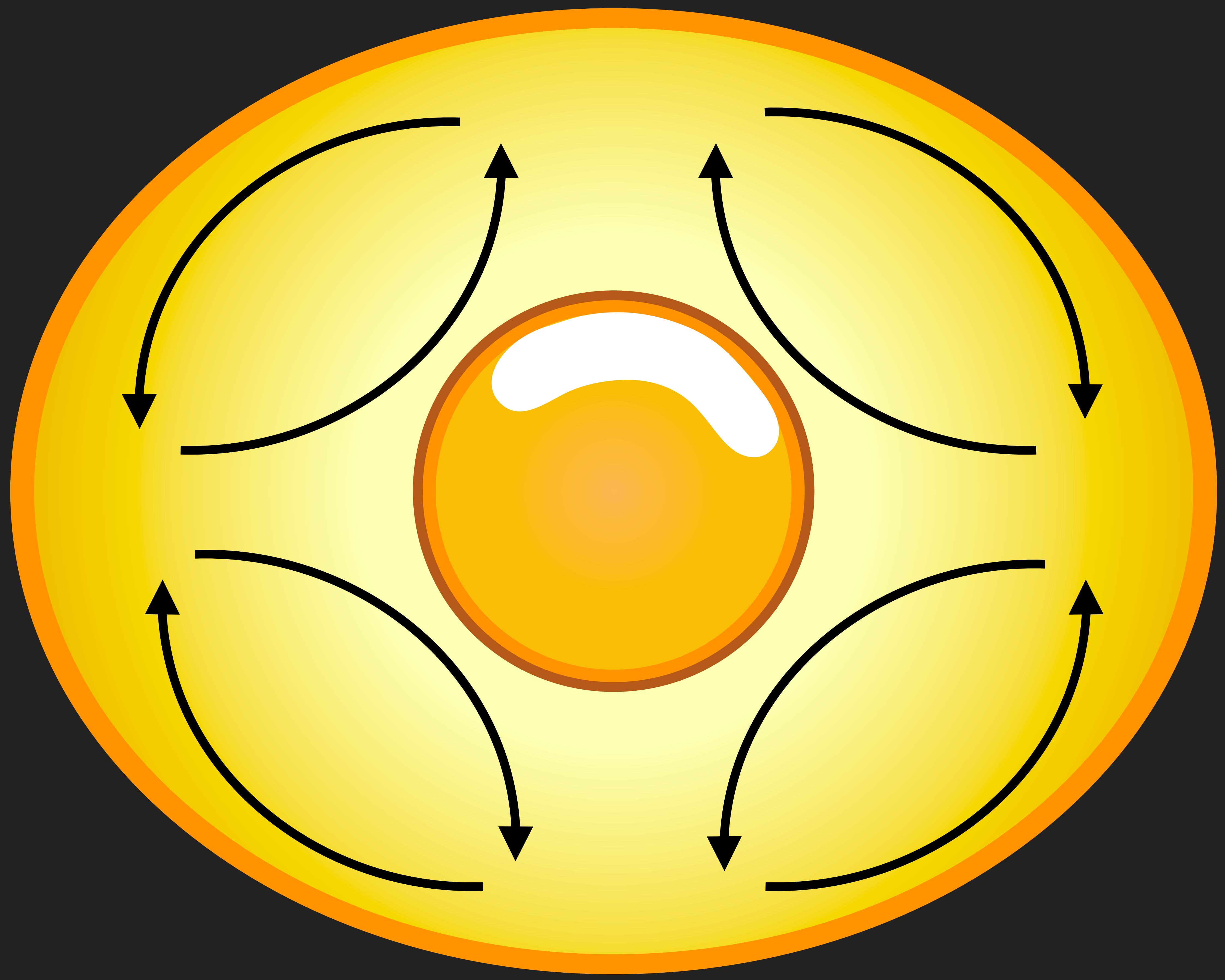
- Large scale baroclinic flows occur to create thermal equilibrium.

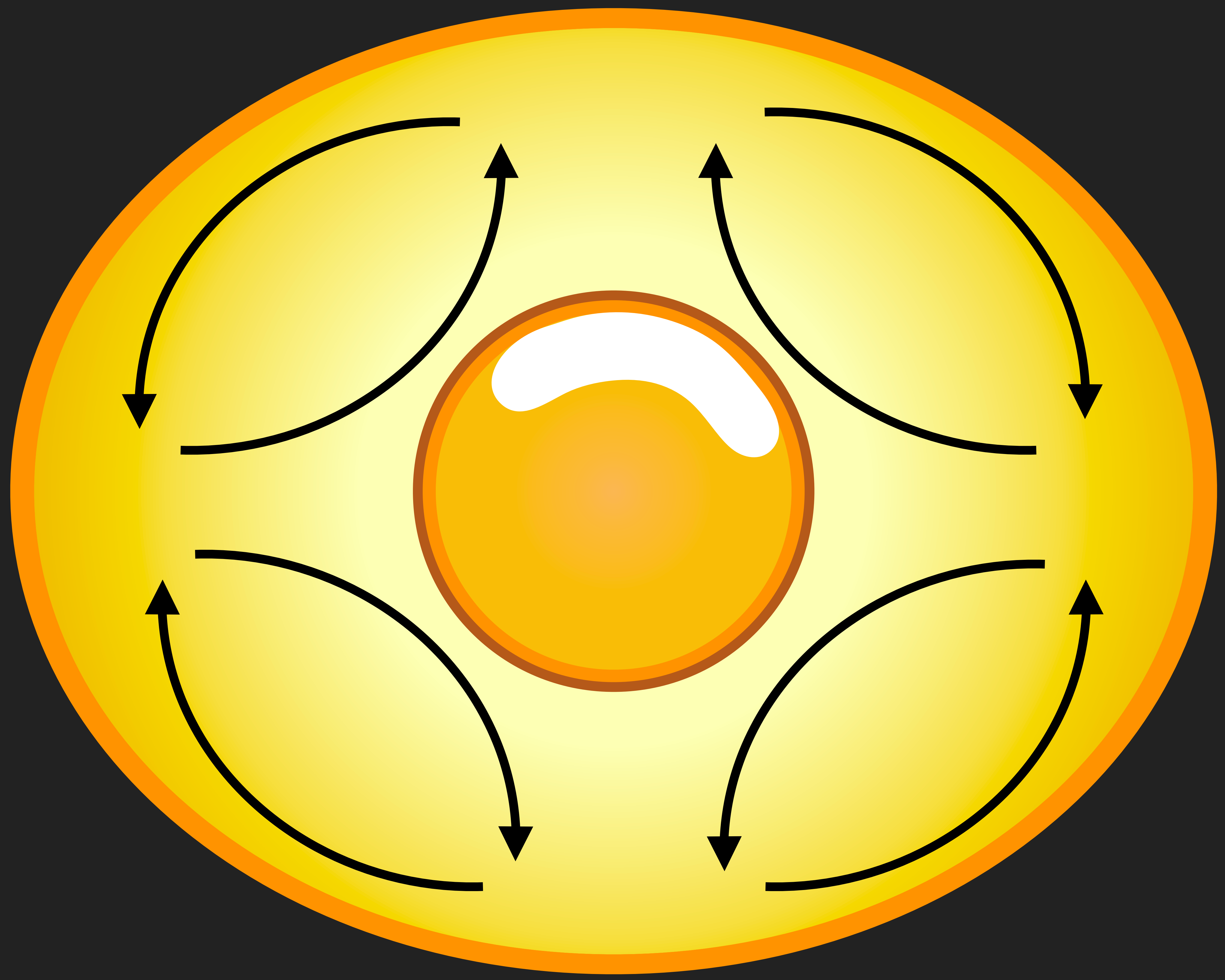


```
&controls

am_nu_ES_factor
D_ES_factor
```

EDDINGTON-SWEET (ES) CIRCULATION

- Rotating star cannot be simultaneously be in hydrostatic and thermal equilibrium.
- Large scale baroclinic flows occur to create thermal equilibrium.
- This picture is wrong though! More on this later..



```
&controls

am_nu_ES_factor
D_ES_factor
```

SPRUIT-TAYLER (ST) DYNAMO

Spruit (2002); Heger et al. (2005)

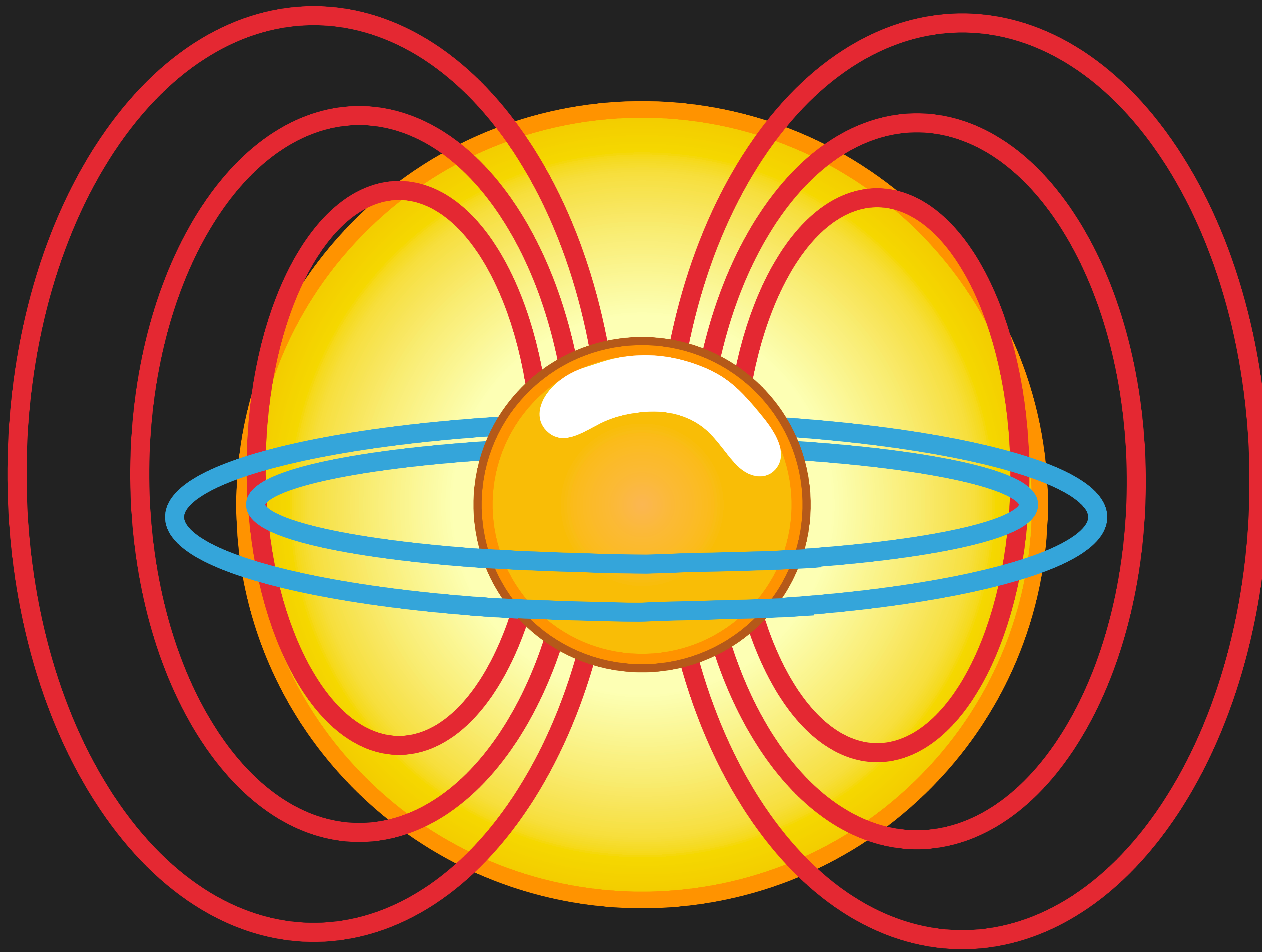
differential rotation amplifies toroidal field from poloidal field

↓

critical B-field → Tayler instability

↓

dynamo regenerates poloidal field



- Saturation when B_ϕ is strong enough to sustain instability, but still allow differential rotation.

```
&controls

am_nu_ST_factor
D_ST_factor
```

ANGULAR MOMENTUM (AM) TRANSPORT IN MESA

- MESA AM transport is fully diffusive.

$$\left(\frac{\partial \Omega}{\partial t}\right)_m = \frac{1}{i}\left(\frac{\partial}{\partial m}\right)_t \left[(4\pi r^2 \rho)^2 i \nu_{\mathrm{AM}} \left(\frac{\partial \Omega}{\partial m}\right)_t\right] - \frac{\Omega}{r}\left(\frac{\partial r}{\partial t}\right)_m \left(\frac{\mathrm{d}\ln i}{\mathrm{d}\ln r}\right)$$

$\nu_{\mathrm{AM}} =$

```
am_nu_non_rotation_factor * am_nu_non_rot +
am_nu_factor * am_nu_rot
```

```
set_uniform_am_nu_non_rot = .true.
uniform_am_nu_non_rot = <>
```

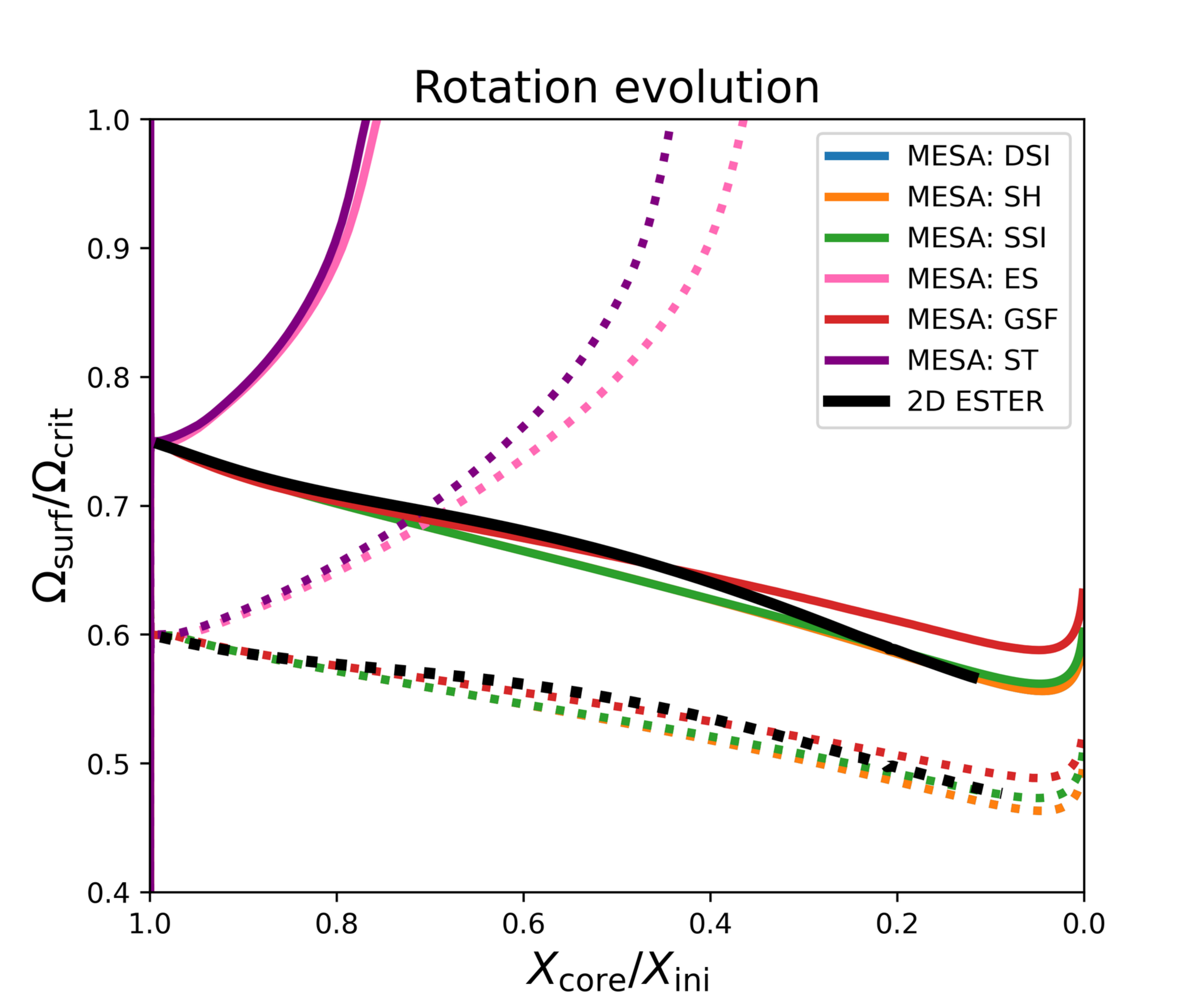
See Heger et al. (2000)

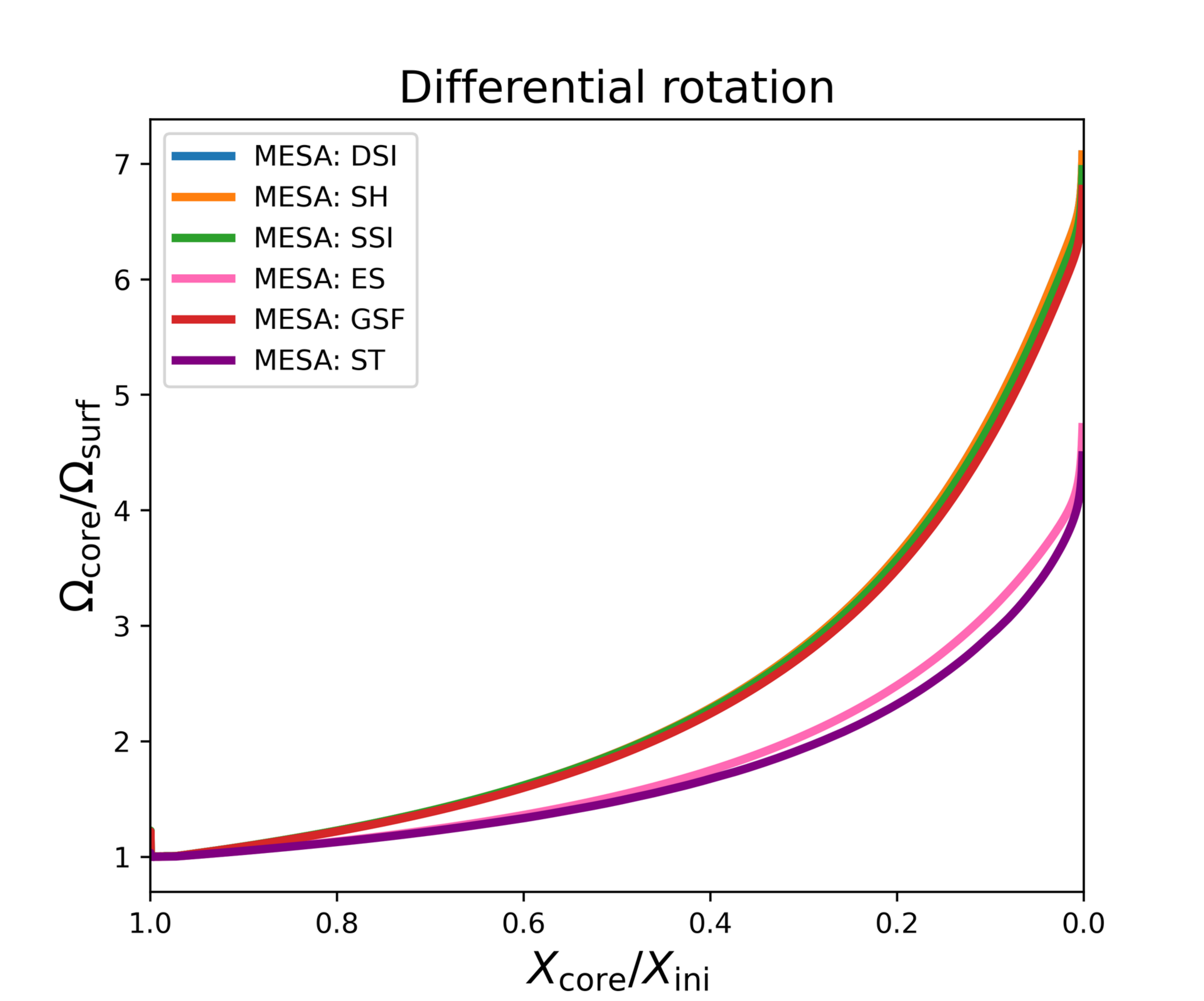
```
am_nu_rot =
am_nu_DSI_factor * D_DSI +
am_nu_SH_factor  * D_SH  +
am_nu_SSI_factor * D_SSI +
am_nu_ES_factor  * D_ES  +
am_nu_GSF_factor * D_GSF +
am_nu_ST_factor  * D_ST
```

LAB 1: ROTATION AND AM TRANSPORT

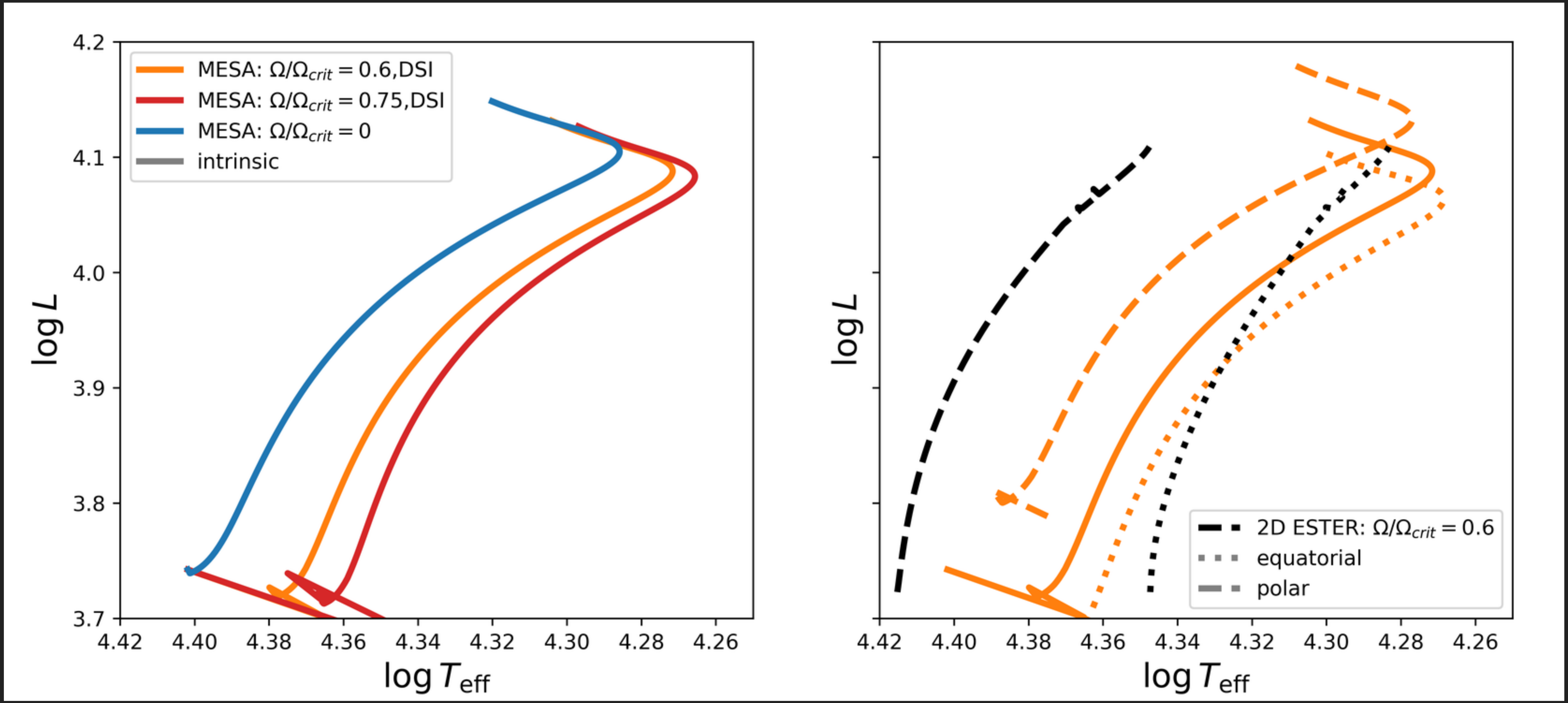
Lead TA: Beatriz Bordadágua

LAB 1 RECAP


Rotation evolution
1.0
0.9
0.8
0.7
0.6
0.5
0.4
$\Omega_{surf}/\Omega_{crit}$
1.0
0.8
0.6
0.4
0.2
0.0
X_{core}/X_{ini}
MESA: DSI
MESA: SH
MESA: SSI
MESA: ES
MESA: GSF
MESA: ST
2D ESTER


Differential rotation
7
6
5
4
3
2
1
$\Omega_{core}/\Omega_{surf}$
1.0
0.8
0.6
0.4
0.2
0.0
X_{core}/X_{ini}
MESA: DSI
MESA: SH
MESA: SSI
MESA: ES
MESA: GSF
MESA: ST

LAB 1 RECAP



MESA: $\Omega/\Omega_{crit} = 0.6$,DSI
MESA: $\Omega/\Omega_{crit} = 0.75$,DSI
MESA: $\Omega/\Omega_{crit} = 0$
intrinsic
log L
log T_{eff}
4.2
4.1
4.0
3.9
3.8
3.7
4.42
4.40
4.38
4.36
4.34
4.32
4.30
4.28
4.26
2D ESTER: $\Omega/\Omega_{crit} = 0.6$
equatorial
polar

ESTER: ÉVOLUTION STELLAIRE EN ROTATION

- Spectral methods to solve stellar structure equations in 2D.

Chebychev polynomials

$$\rho(\zeta, \theta) = \sum_{i}^{n_r-1} \sum_{j}^{n_\theta-1} \rho_{ij} T_i(\zeta) P_j(\cos\theta)$$

Legendre polynomials

ESTER

Espinosa Lara & Rieutord 2013; Rieutord et al. 2016

Spectral methods

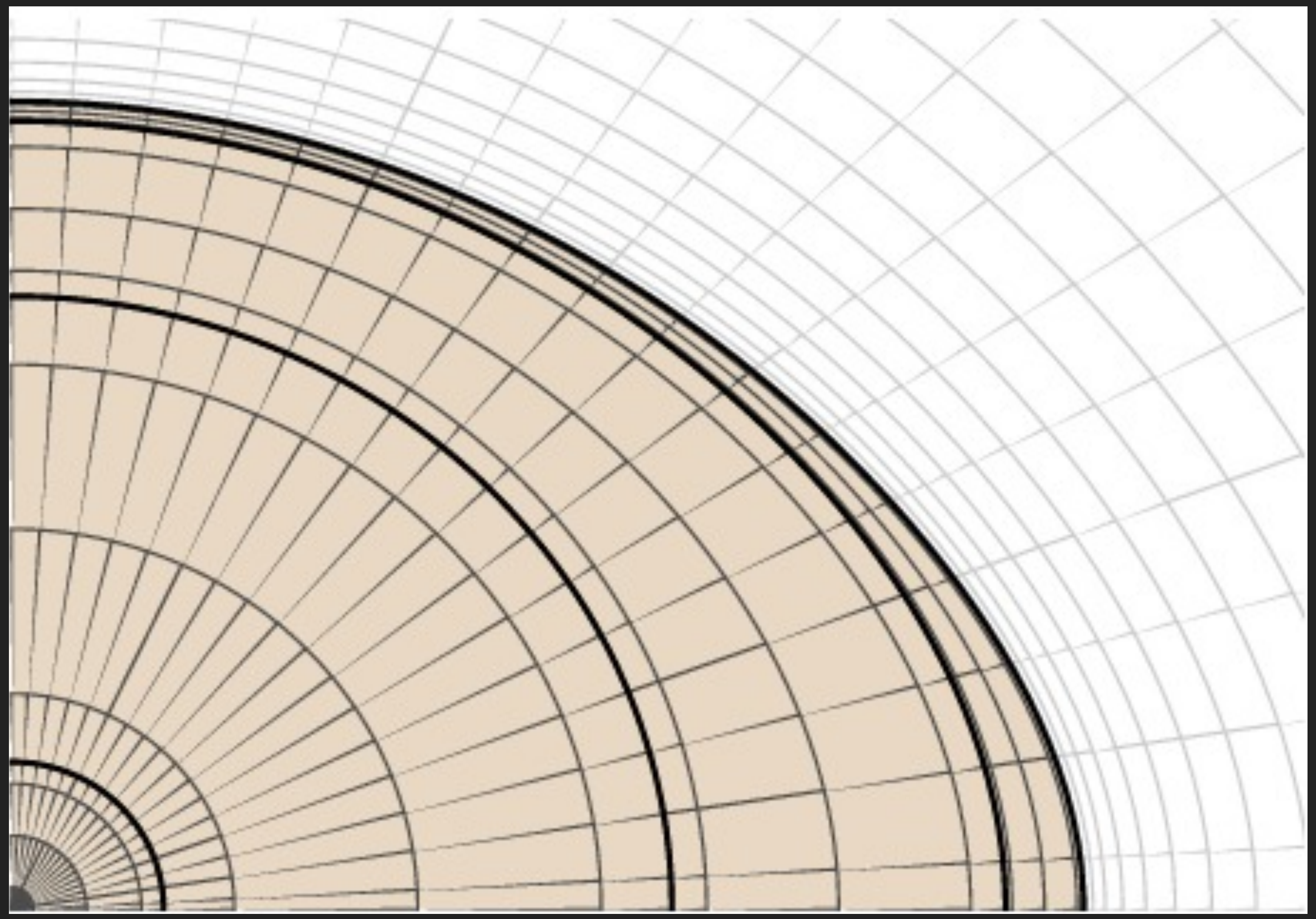
$$\partial_\zeta \rho(\zeta_j, \theta) \approx \sum_k D_{jk} \rho(\zeta_k, \theta)$$

Finite differences (MESA)

$$\partial_x \rho(x_i) \approx \frac{\rho(x_{i+1}) - \rho(x_i)}{x_{i+1} - x_i}$$

ESTER: ÉVOLUTION STELLAIRE EN ROTATION

- Spectral methods to solve stellar structure equations in 2D.
- Centrifugal deformation included.



ESTER

Espinosa Lara & Rieutord 2013; Rieutord et al. 2016

- Velocity field computed self-consistently (differential rotation and meridional circulation).
- Publicly available at `https://ester-project.github.io/ester/`

CHEMICAL AND ANGULAR MOMENTUM TRANSPORT

- First full 2-D stellar evolution models.

$$\frac{\mathrm{d}X}{\mathrm{d}t} + \mathbf{v} \cdot \nabla X = \frac{1}{\rho} \nabla(\rho[D] \nabla X) + \dot{X}_{\mathrm{nuc}}$$

- Angular momentum transport

$$\frac{\partial(s^2\Omega)}{\partial t} + \mathbf{v} \cdot \nabla(s^2\Omega) = \frac{1}{\rho} \nabla(\rho \nu s^2 \nabla \Omega)$$

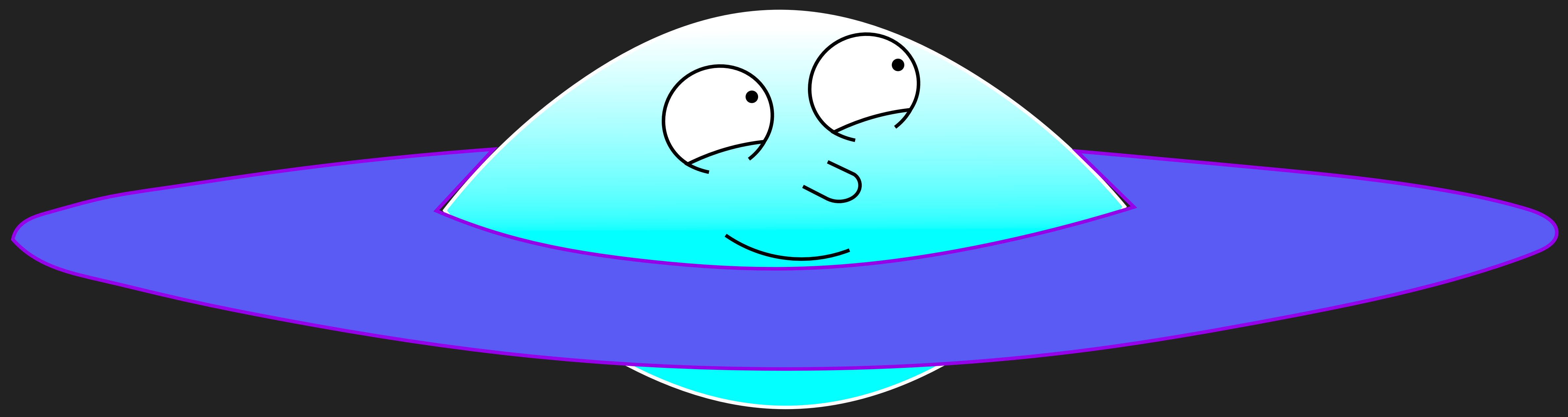
distance to rotation axis

velocity field

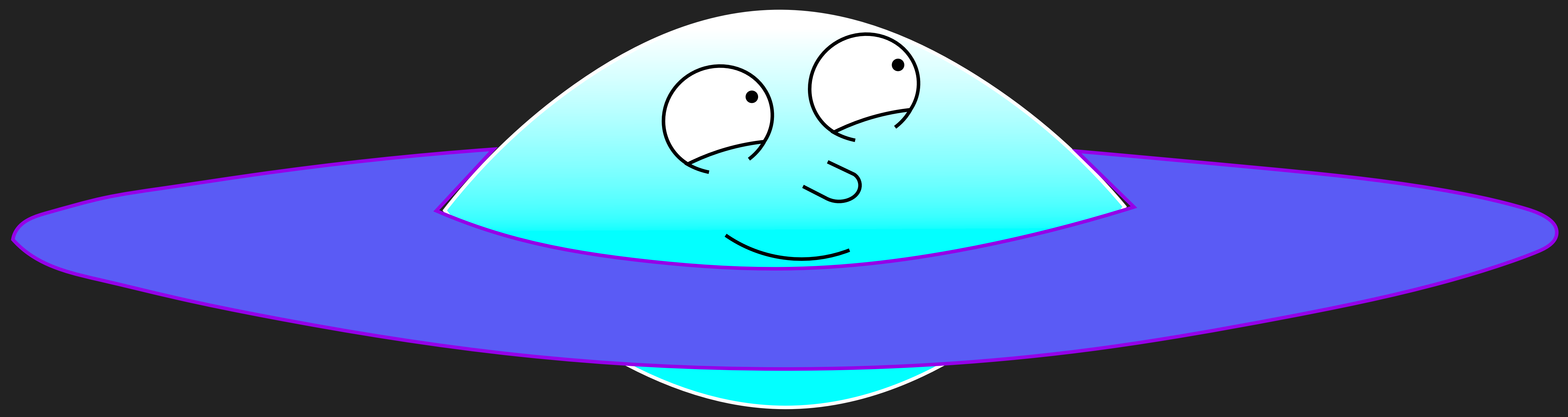
turbulent viscosity

- No magnetism or waves (yet).

BE STARS

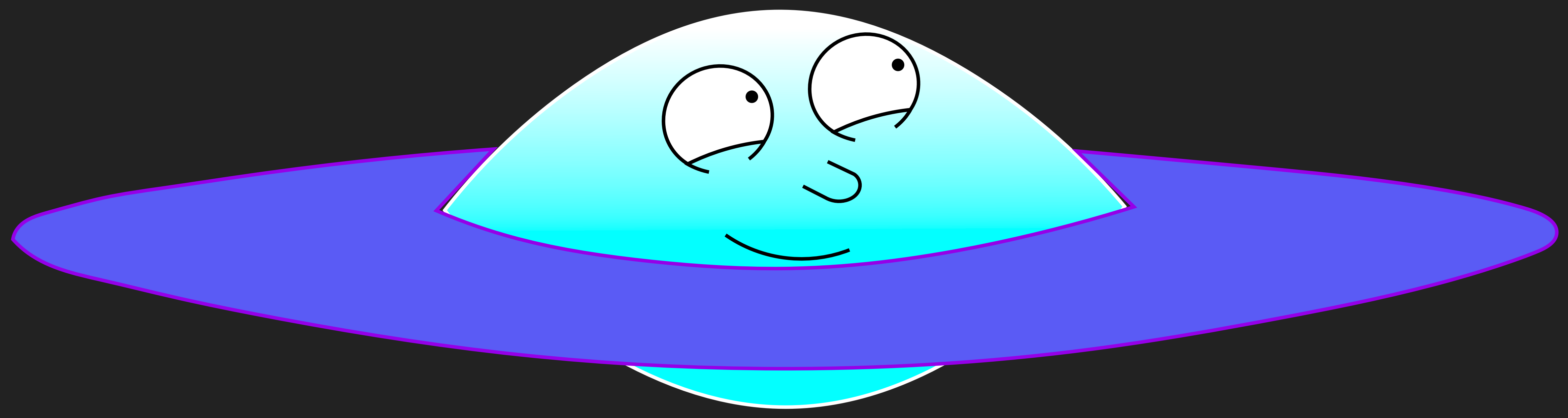


BE STARS



- Stars of spectral type B that are rotating sufficiently fast to form a decretion disk.

BE STARS

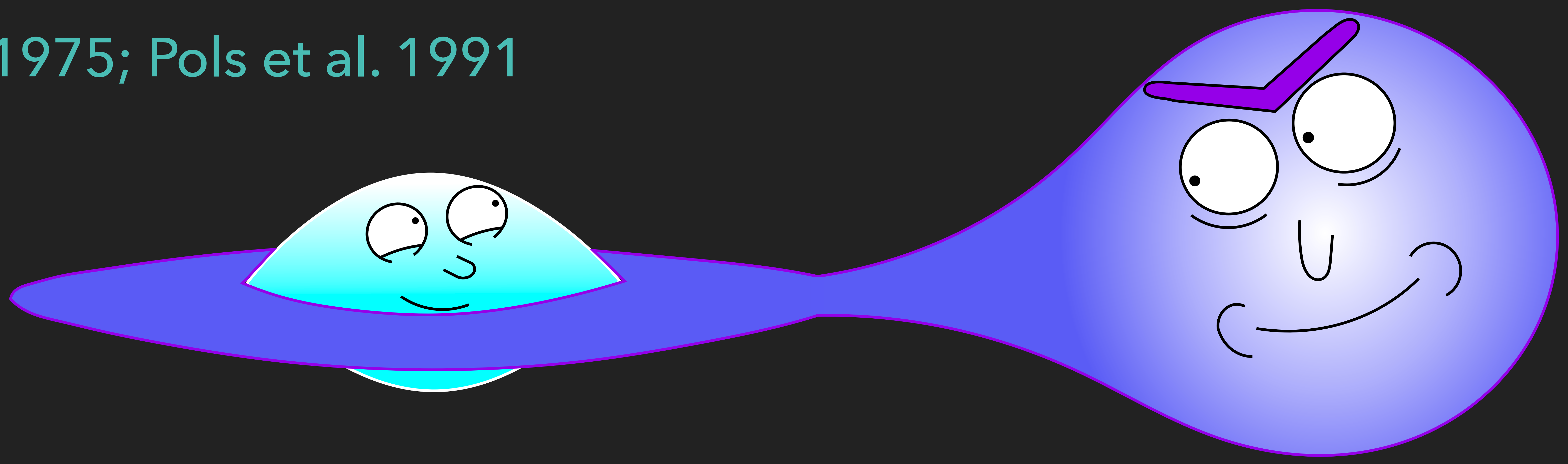


- Stars of spectral type B that are rotating sufficiently fast to form a decretion disk.
- Debated what fraction of critical rotation is needed to expel matter.
  (e.g. Cranmer 2005; Frémat et al. 2005)

BE STAR FORMATION

Binary channel

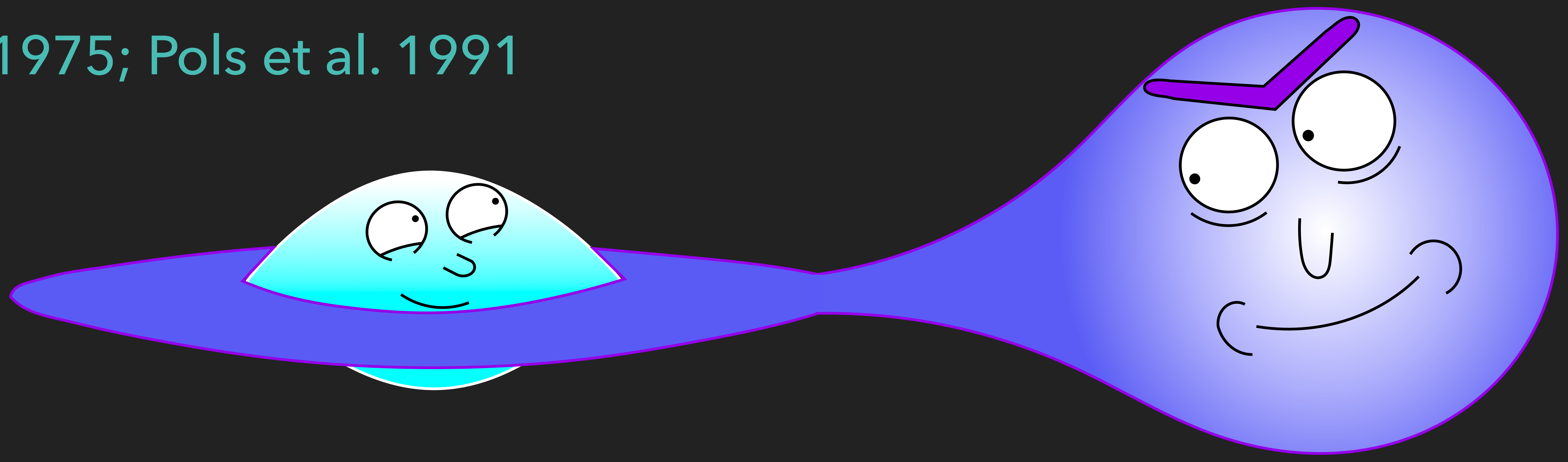
e.g. Kriz & Harmanec 1975; Pols et al. 1991



BE STAR FORMATION

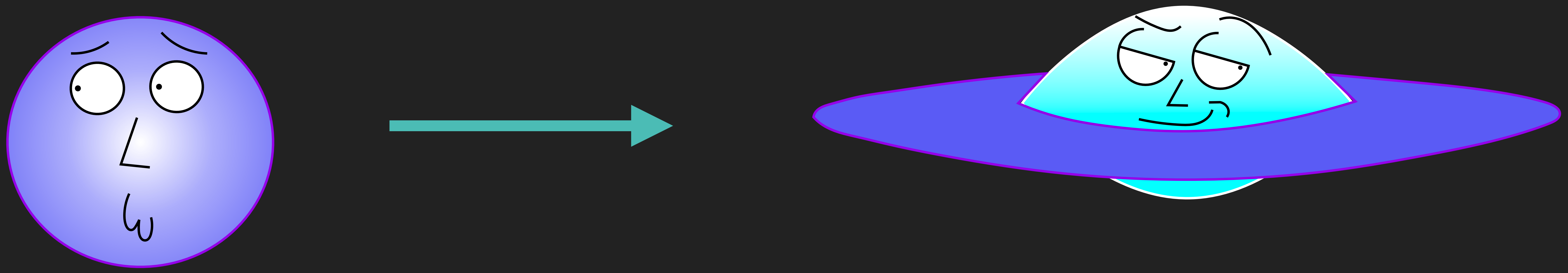
Binary channel

e.g. Kriz & Harmanec 1975; Pols et al. 1991



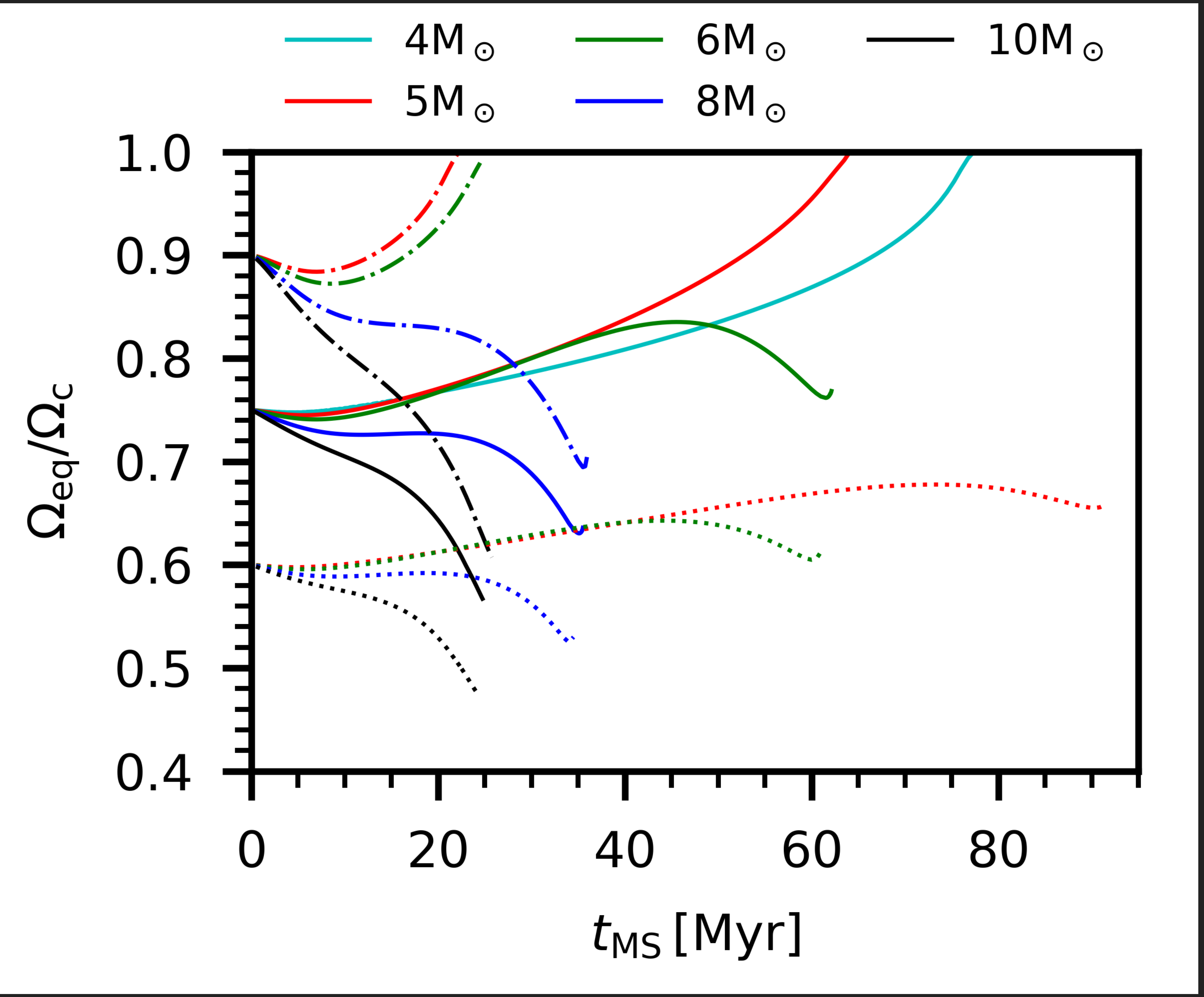
Single-star channel

e.g. Bodenheimer 1995; Ekström et al. (2008); Granada. et al. (2013); Hastings et al. (2020)



ROTATIONAL EVOLUTION

Mombarg, Rieutord & Espinosa Lara (2024, A&A, 683, A94)



fraction critical rotation

main-sequence age

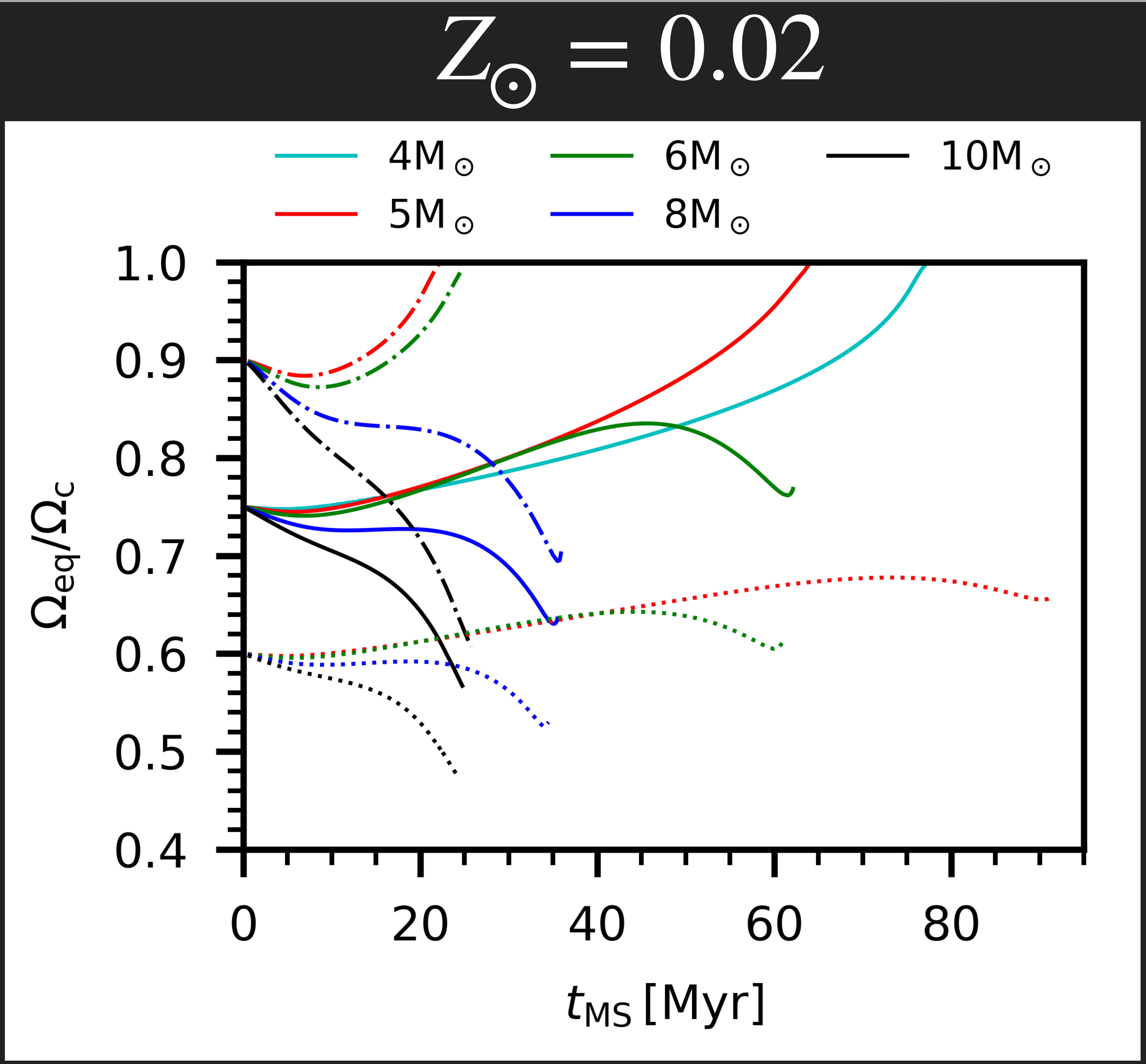
$Z_\odot$

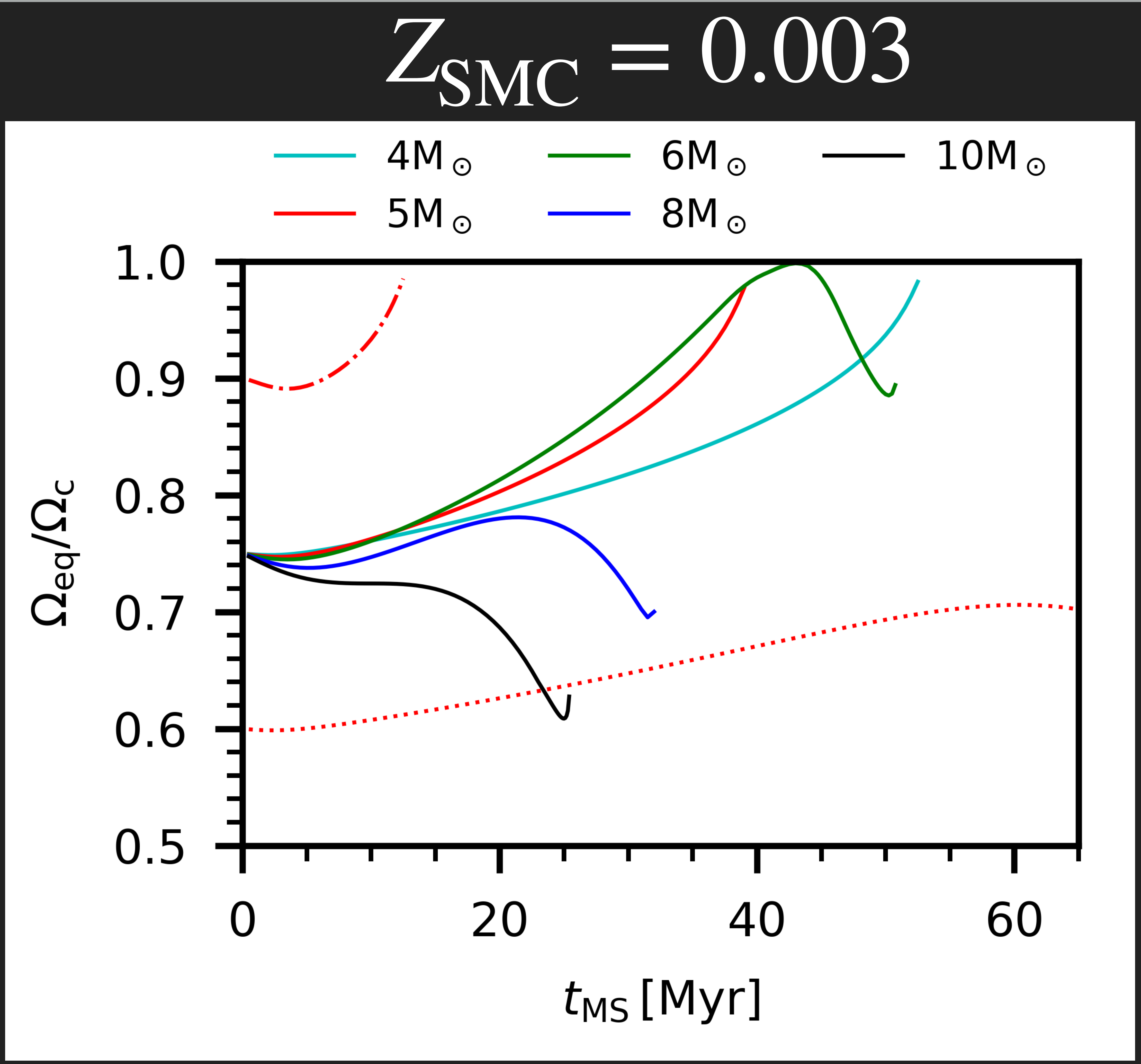
(Pre-main sequence lifetimes between 4.4 and 0.34 Myr.)

- Lower-mass stars increase their fraction of critical rotation, higher-mass stars do not.

METALLICITY EFFECT

Mombarg, Rieutord & Espinosa Lara (2024, A&A, 683, A94)

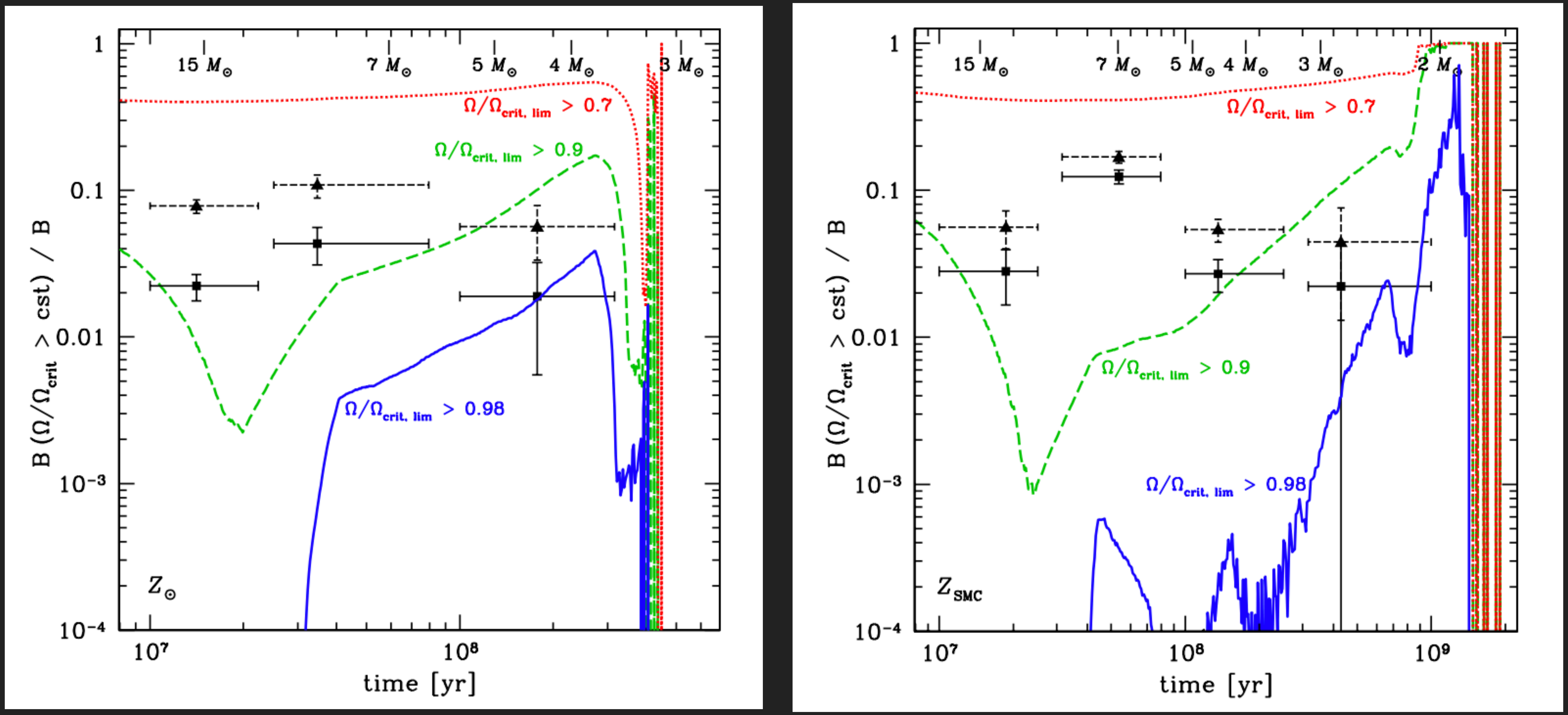




- Lower metallicity stars spin up faster, and mass threshold moves up.

OBSERVED BE FRACTIONS

- Peak in fraction Be/(Be+B) during evolution seen in SMC stars.



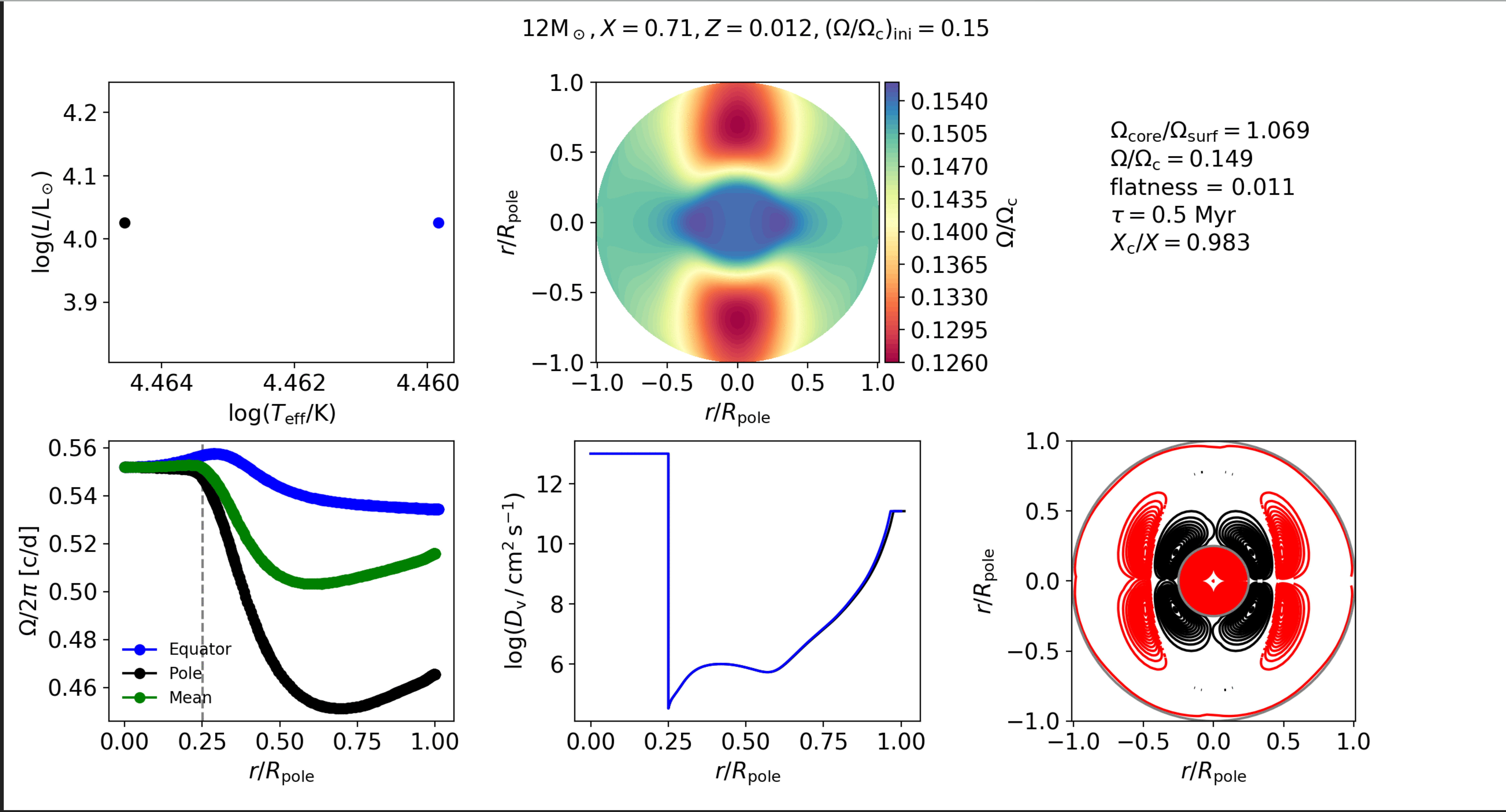
Figures from Granada et al. (2013)

Data points from McSwain & Gies (2005) and Martayan et al. (2010)

- Metallicity dependence not predicted by 1D models Ekström et al. (2008,2012); Granada et al. (2013). In models of Hastings et al. (2020) effect caused by mass loss.

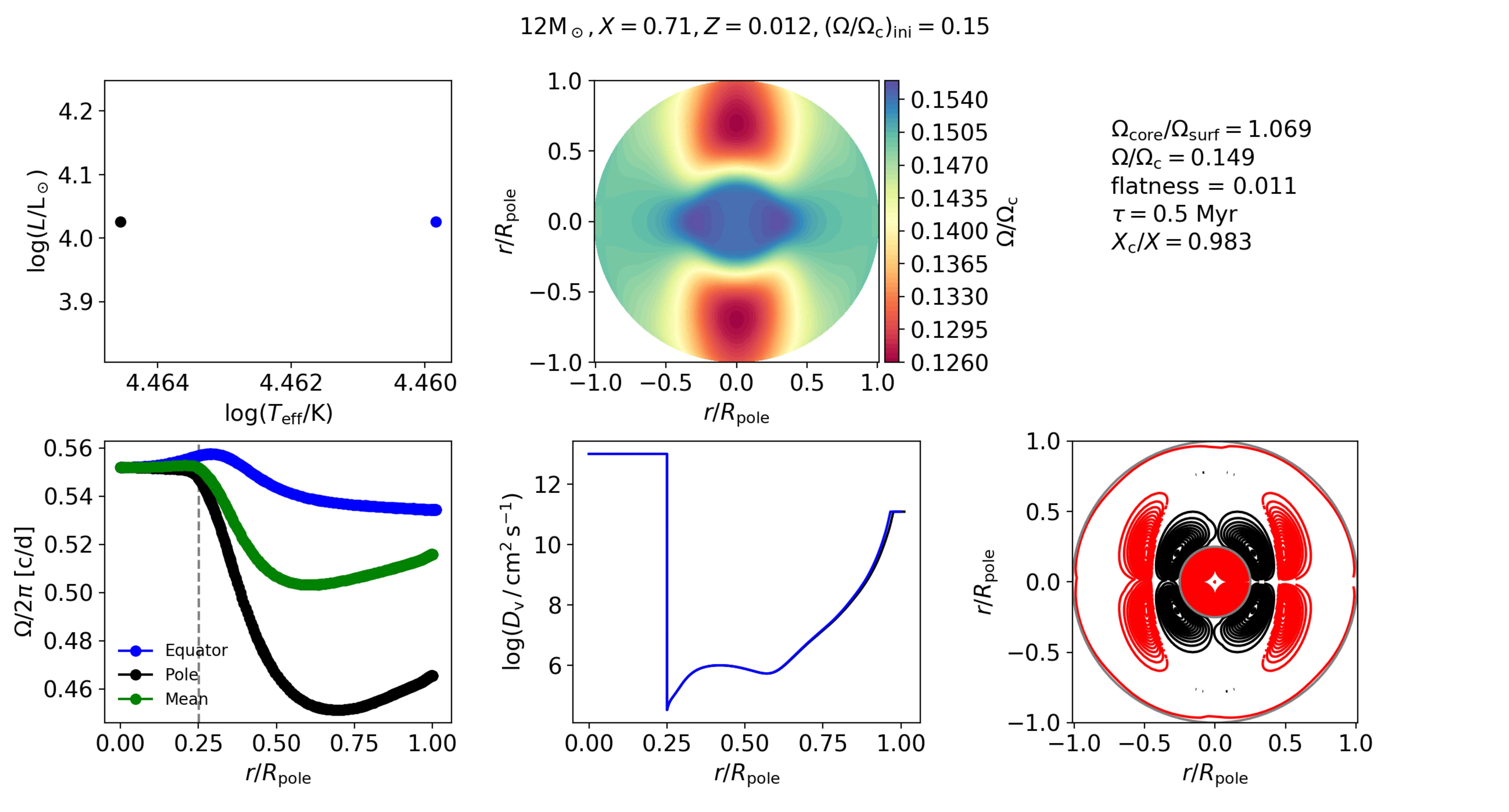
THE FIRST 2D EVOLUTION MODELS

Mombarg et al. (2023), A&A, 677, L5



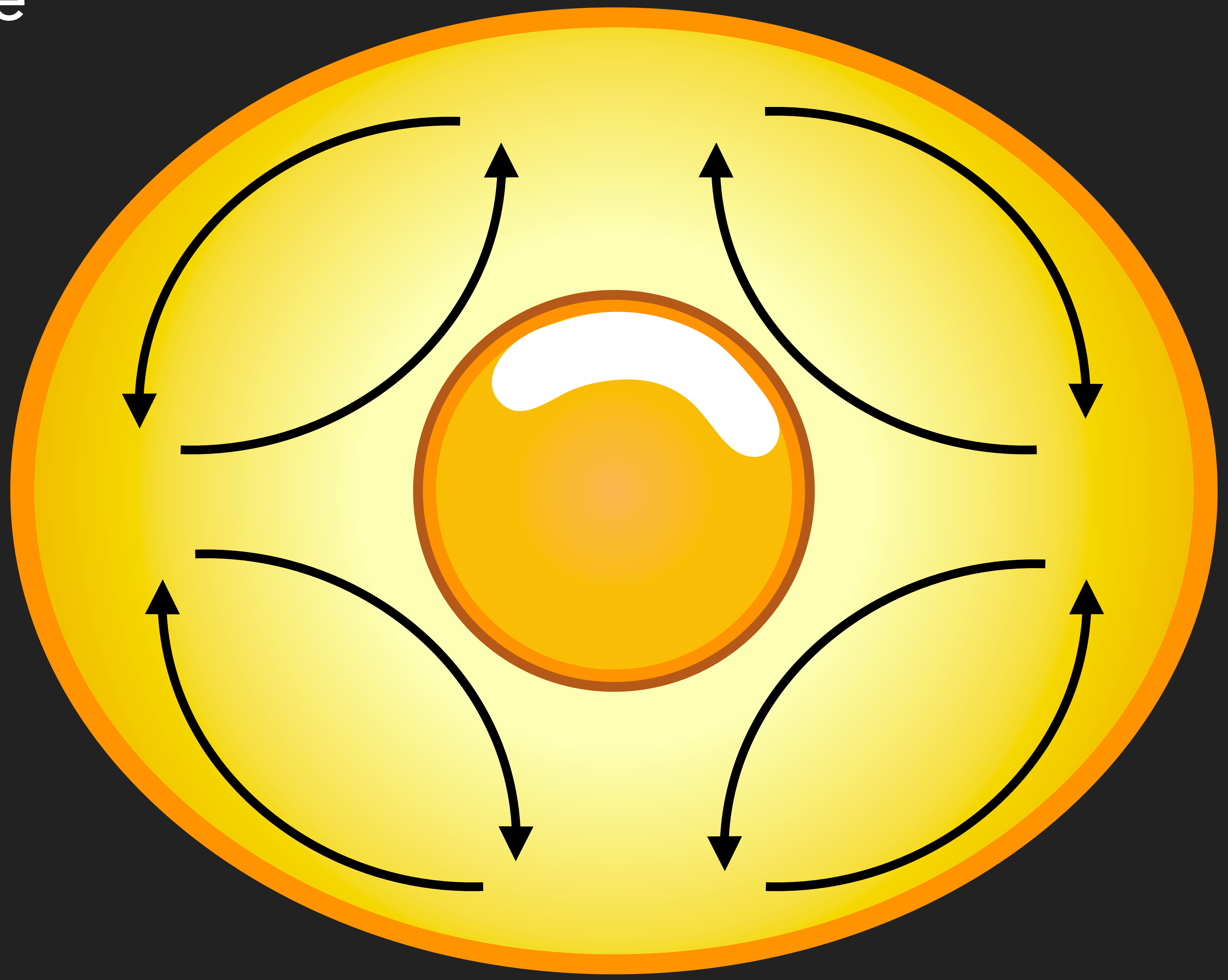
THE FIRST 2D EVOLUTION MODELS

Mombarg et al. (2023), A&A, 677, L5

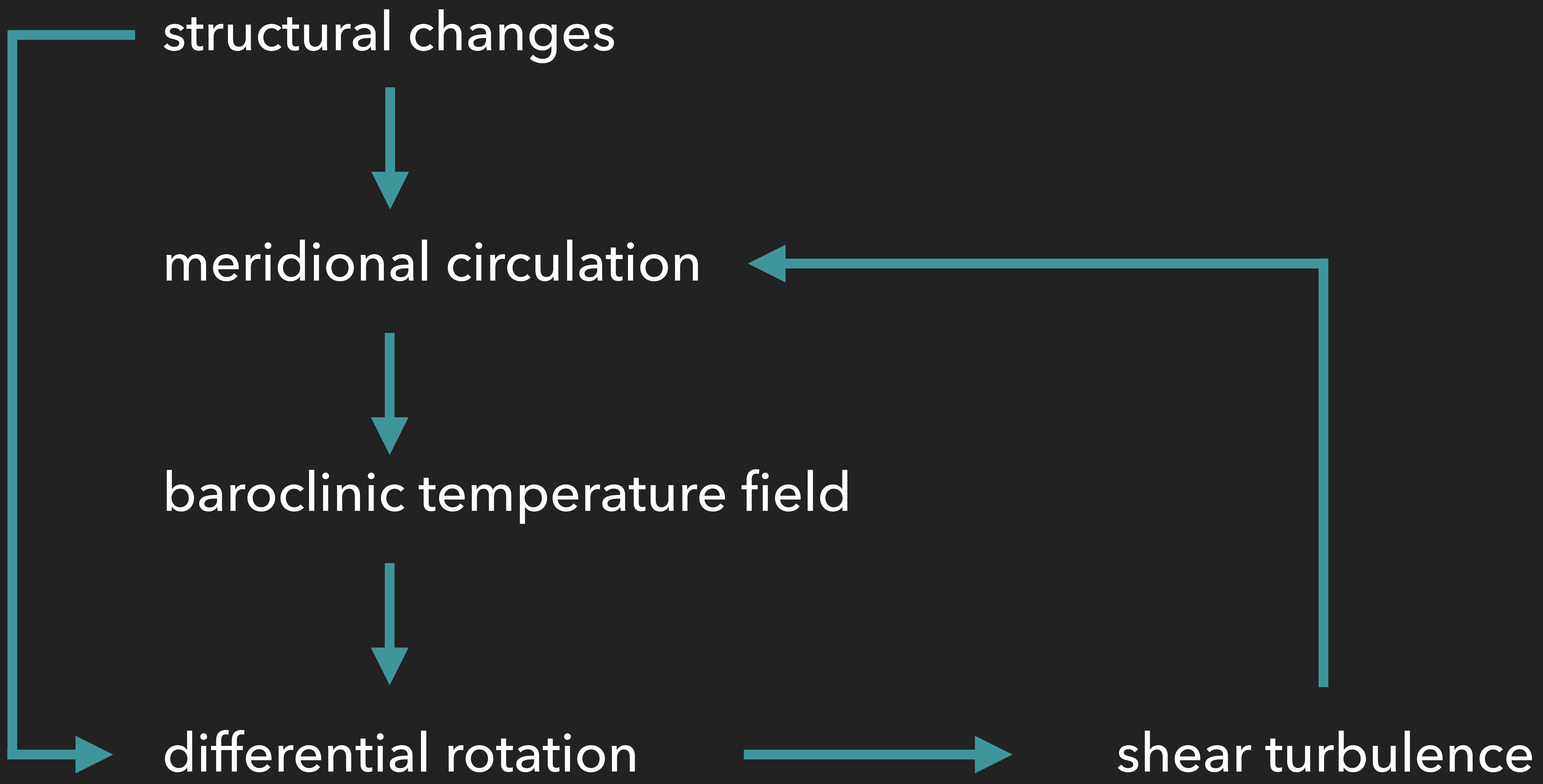


EDDINGTON-SWEET CIRCULATION

- Rotating star cannot be simultaneously be in hydrostatic and thermal equilibrium.
- Large scale baroclinic flows occur to create thermal equilibrium.
- This picture is wrong!
  Meridional circulation disappears when viscosity is zero.



THE TRANSPORT LOOP



Rieutord (2006)
Decressin et al. (2009)

EDDINGTON-SWEET CIRCULATION IN MESA

- Eq. (35) of Heger et al. (2000):

$$v_\mathrm{e} = \frac{\nabla_\mathrm{ad}}{\delta(\nabla_\mathrm{ad} - \nabla)} \frac{\Omega^2 r^3 l}{(Gm)^2} \left[\frac{2(\epsilon_\mathrm{n} + \epsilon_\nu) r^2}{l} - \frac{2r^2}{m} - \frac{3}{4\pi\rho r} \right]$$

- Stabilising effect of chemical gradient ∇_μ (Mestel 1952, 1953)

$$v_\mathrm{ES} = \max\left(|v_e| - |v_\mu|, 0 \right) \qquad v_\mu = \frac{H_P}{\tau^*_\mathrm{KH}} \frac{\varphi \nabla_\mu}{\delta(\nabla - \nabla_\mathrm{ad})}$$

$$\tau^*_\mathrm{KH} = \frac{Gm^2}{r(l - m\epsilon_\nu)}$$

$$\varphi \equiv \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_{\mu,P}$$

$$\delta \equiv \frac{\chi_T}{\chi_\rho}$$

ADDING A PROFILE COLUMN

```
integer function how_many_extra_profile_columns(id)

   integer, intent(in) :: id
   integer :: ierr
   type (star_info), pointer :: s
   ierr = 0
   call star_ptr(id, s, ierr)
   if (ierr /= 0) return

   how_many_extra_profile_columns = 1

end function how_many_extra_profile_columns



integer function data_for_extra_profile_columns(id, n, nz, names, vals, ierr)
   …

   names(1) = ‘my_extra_column’

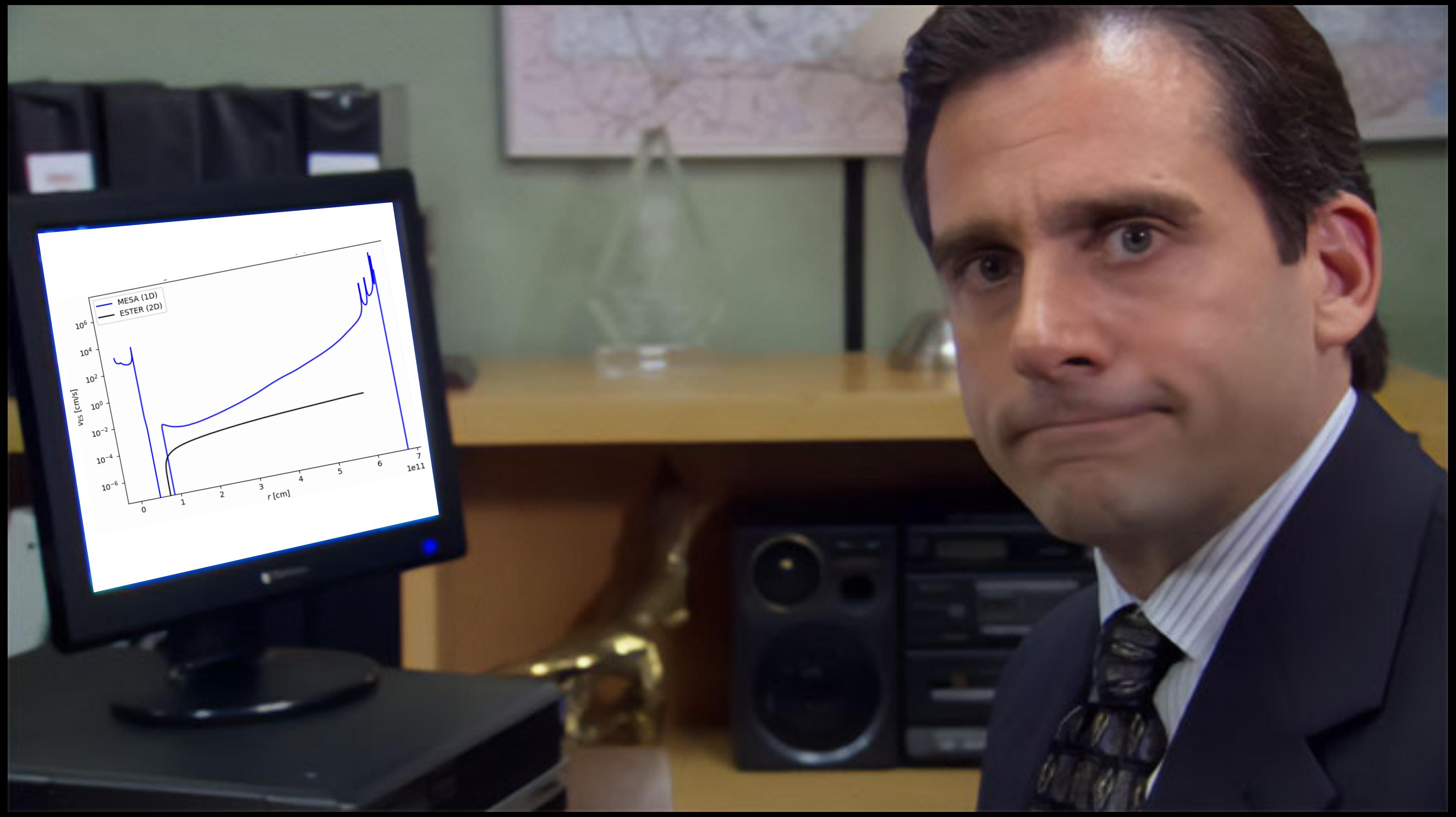
   do k = 1, nz
     vals(k,1) = …
   end do

end function data_for_extra_profile_columns
```

Don’t forget to recompile after you’ve changed the run_star_extras.f90!

```
./clean; ./mk
```

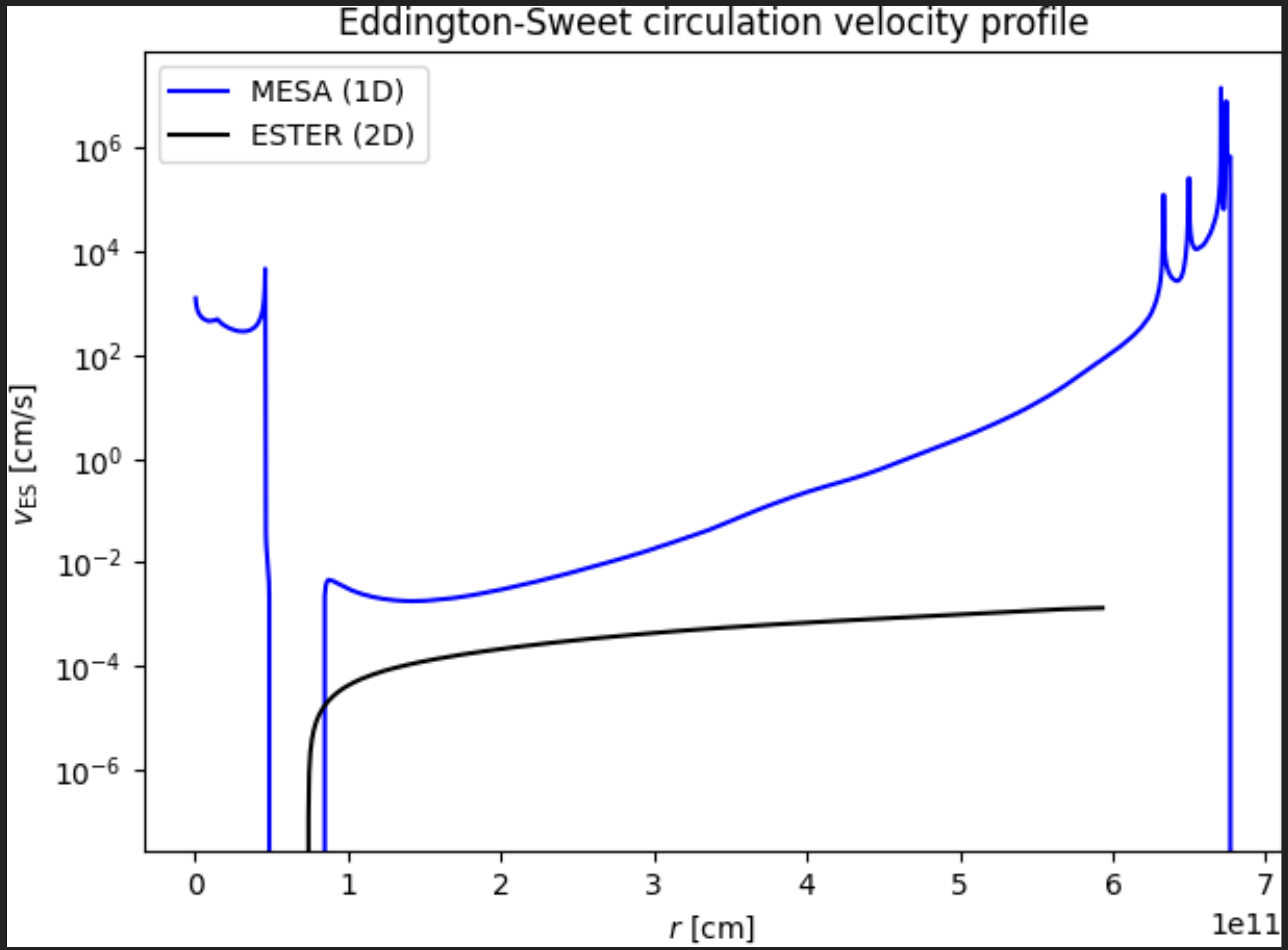
LAB 2: COMPARING MERIDIONAL CIRCULATION - 1D VS 2D



Lead TA: Philip Mocz

LAB 2 RECAP

- ES circulation velocity predicted by MESA much higher than the meridional circulation velocity predicted by 2D models.



ANGULAR MOMENTUM (AM) TRANSPORT

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\Omega\right) = \frac{1}{r^2}\partial_r\left(\rho\nu r^4\partial_r\Omega\right) + \frac{1}{5r^2}\partial_r\left(\rho r^4\Omega U\right) - \frac{1}{r^2}\partial_r\left(r^2F_J(r)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \dot{r}\partial_r$$

diffusion

advection

internal gravity waves

CHEMICAL TRANSPORT (MIXING)

mass fraction of species i

$$\frac{\partial X_i}{\partial t} = \frac{\partial}{\partial m}\left(\left[4\pi\rho r^2\right]^2 [D_{\mathrm{marco}} + D_{\mathrm{micro},i}] \frac{\partial X_i}{\partial m} \right)$$

can be set to ν_{AM} to do rotational mixing

diffusion

ROTATIONAL MIXING OF CHEMICAL ELEMENTS

- Option 1: Use the same `D_*_factor` as `am_nu_*_factor`.
  - Not clear if this makes physical sense.
  - Could give noisy N^2 profiles, problem for asteroseismology.
- Option 2: Use the Zahn (1992) and Chaboyer & Zahn (1992) formalism

$$D_\mathrm{v} = \frac{\mathrm{Ri_c}}{N_T^2/(K + D_\mathrm{h}) + N_\mu^2/D_\mathrm{h}} \left(r \frac{\mathrm{d}\Omega}{\mathrm{d}r} \right)^2$$

$$D_\mathrm{h} = \frac{1}{C_\mathrm{h}} r \left| 2V - \alpha U \right|$$

α depends on shear,
1 for uniform rotation

$$D_\mathrm{eff} = \frac{1}{30} \frac{(rU)^2}{D_\mathrm{h}}$$

See e.g.
Keszthelyi et al. (2022)
Mombarg et al. (2024)
for MESA implementations

ROTATIONAL MIXING OF CHEMICAL ELEMENTS

- Option 1: Use the same `D_*_factor` as `am_nu_*_factor`.
  - Not clear if this makes physical sense.
  - Could give noisy N^2 profiles, problem for asteroseismology.
- Option 2: Use the Zahn (1992) and Chaboyer & Zahn (1992) formalism
  - Cannot reproduce near-core rotation frequencies of F-type stars (Ouazzani et al. 2019; Li et al. (2020), Assumptions on PMS physics!)
  - Also highly parameterised.

See e.g.
Keszthelyi et al. (2022)
Mombarg et al. (2024)
for MESA implementations

ANGULAR MOMENTUM TRANSPORT

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\Omega\right) = \underbrace{\frac{1}{r^2}\partial_r\left(\rho\nu r^4 \partial_r \Omega\right)}_{\text{diffusion}} + \underbrace{\frac{1}{5r^2}\partial_r\left(\rho r^4 \Omega U\right)}_{\text{advection}} - \underbrace{\frac{1}{r^2}\partial_r\left(r^2 F_J(r)\right)}_{\text{internal gravity waves}}$$

$$\frac{\mathrm{d}}{\mathrm{dt}} = \partial_t + \dot{r}\partial_r$$

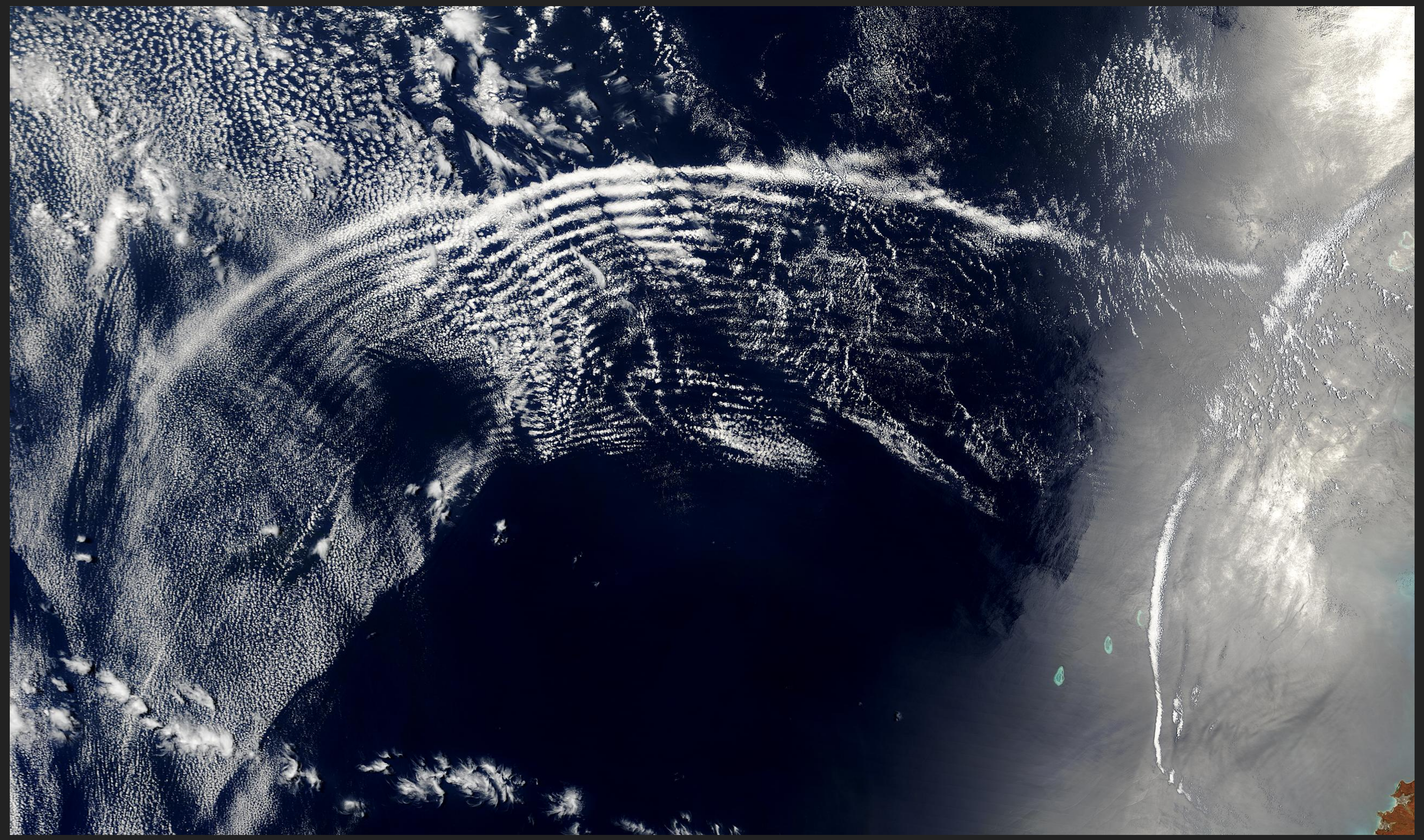
Channels of AM transport

1) radiative damping

2) critical layers

3) wave breaking

e.g. Mathis (2025)



NASA

INTERNAL GRAVITY WAVES (IGW)

- Which frequencies and spherical degrees to include?

$$F_{J,\mathrm{tot}} = \sum_{\omega,\ell,m} F_{J;\ell,m}(\omega)$$

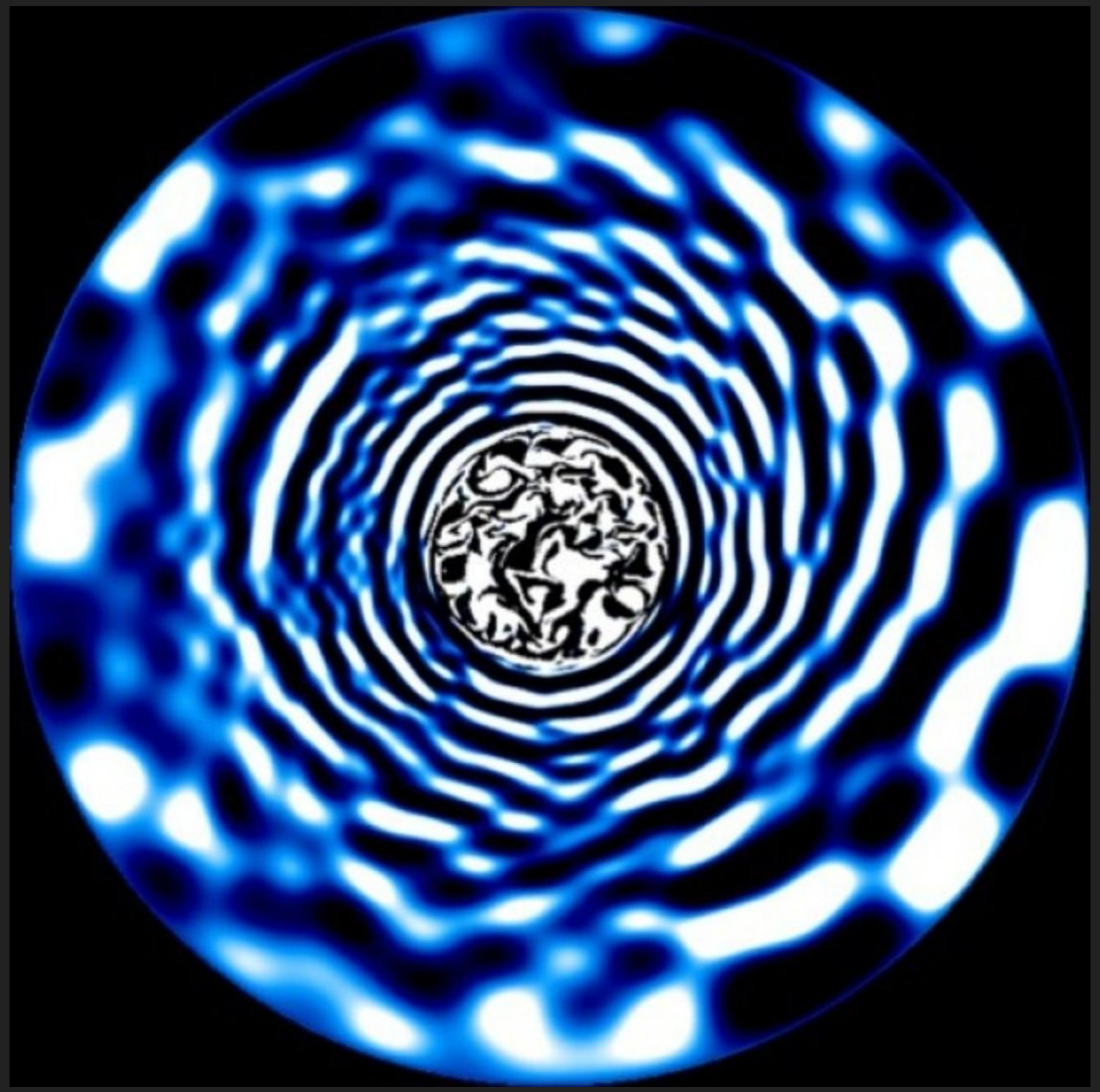
$\propto v^2_{\mathrm{wave};\ell}$

- What is the initial wave velocity?

$$u_{\mathrm{v},0} \propto \omega^{\mathrm{m}} \left(\sqrt{\ell(\ell+1)}\right)^{\mathrm{n}}$$

Spectra	m	n
Flat[F]	0	0
Kumar et al. (1999)[K]	-2.17	1
Lecoanet & Quataert (2013)[LD]	-4.25	2.5
Rogers et al. (2013)[R]	-0.6	-0.9

Table 1 from Ratnasingam et al. (2018)

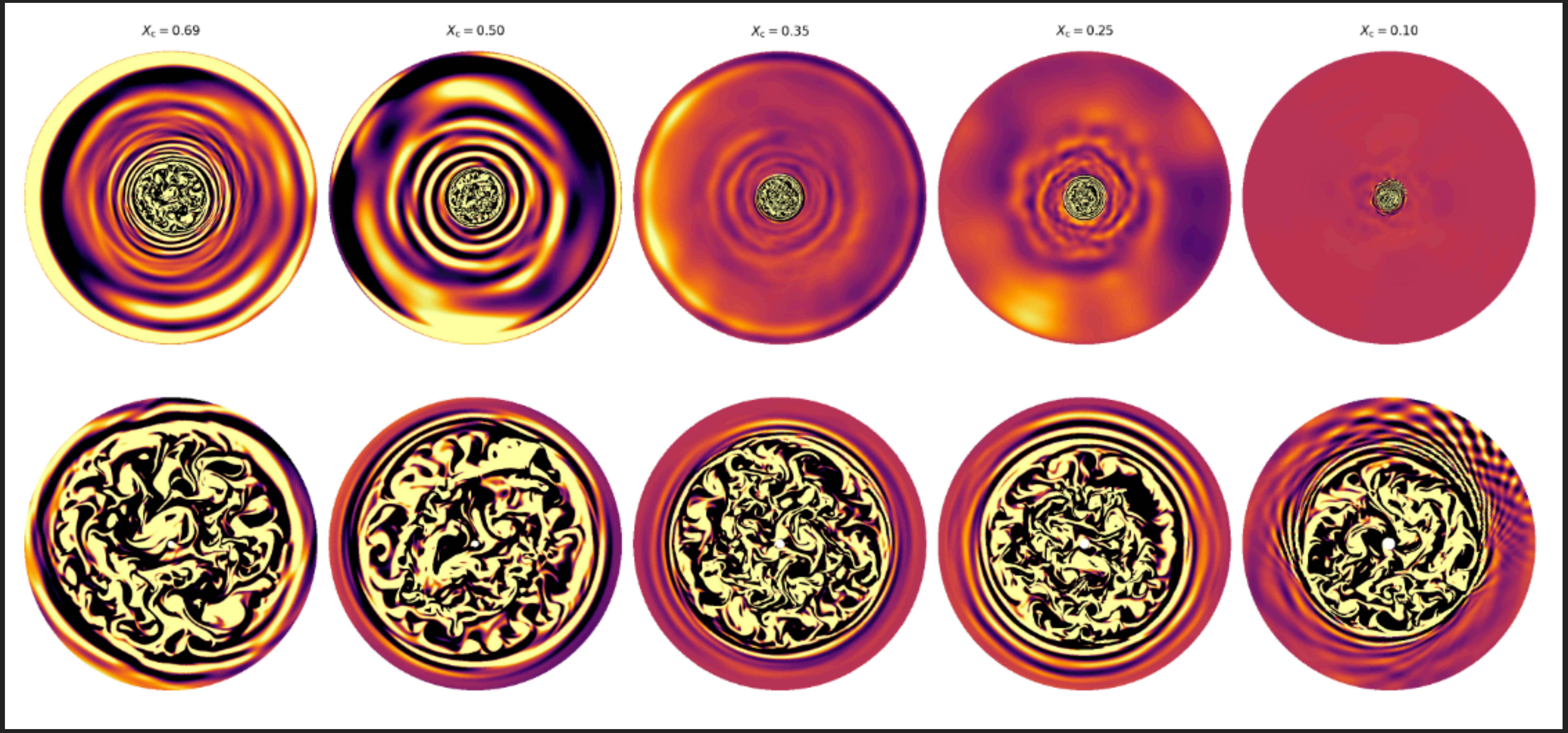


Bowman et al. (2019)

HYDRODYNAMICAL SIMULATIONS OF IGW

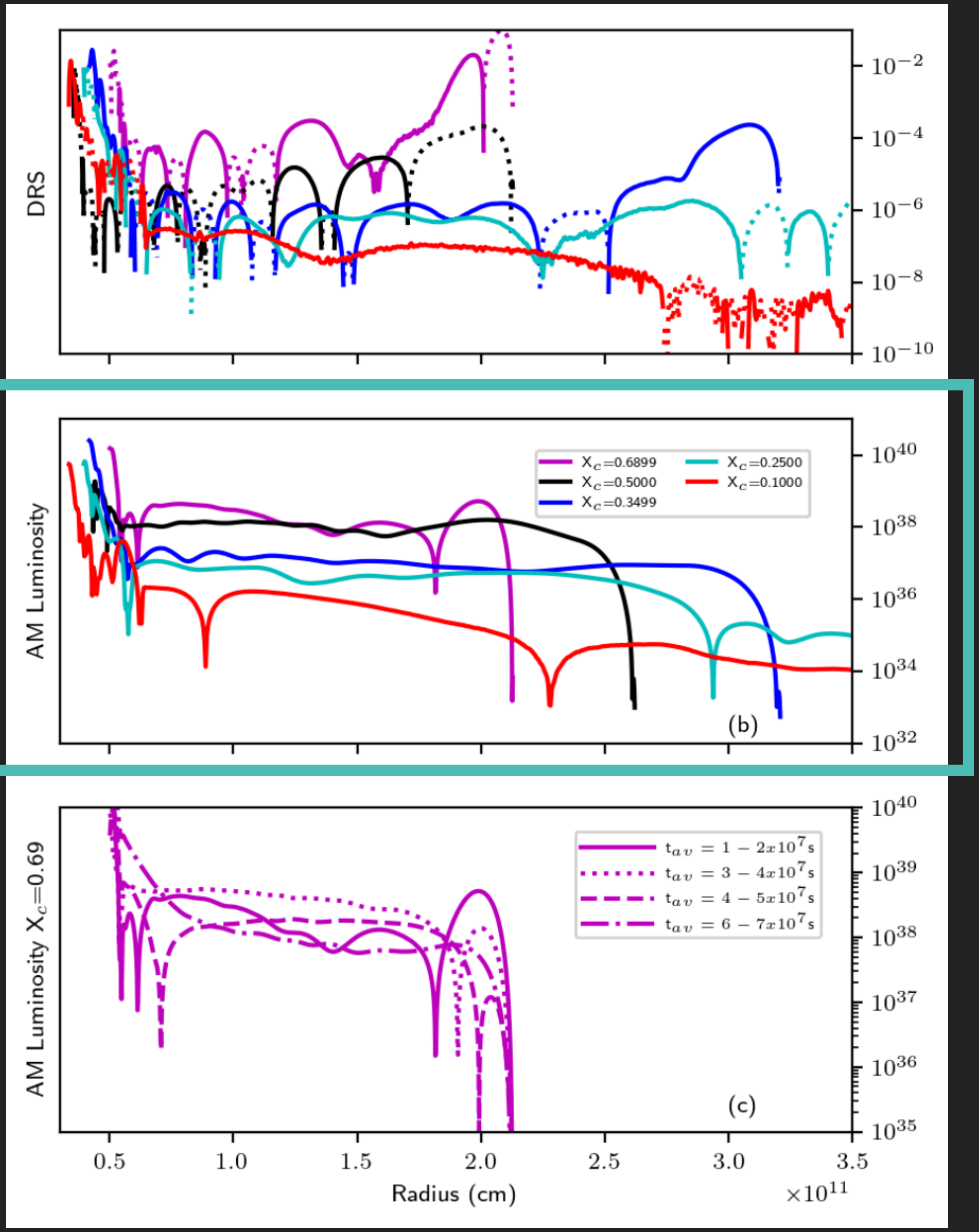
Rogers & Ratnasingam (2025)

age →



- IGWs can transport AM efficiently during the early main sequence.

$$\frac{1}{r^2}\partial_r\left(r^2 F_J(r)\right)$$



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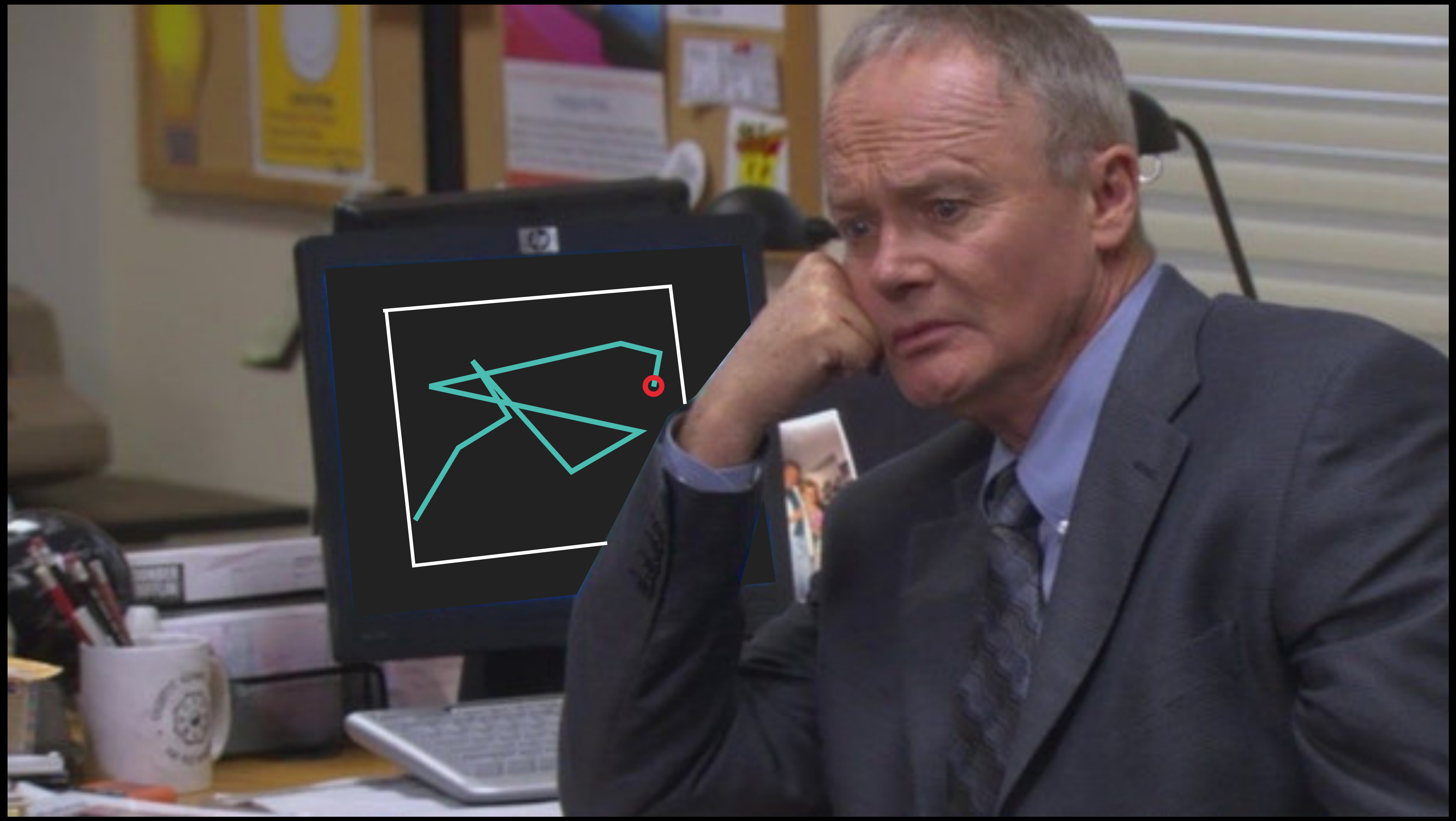
internal gravity waves

ANGULAR MOMENTUM (AM) TRANSPORT

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\Omega\right) = \frac{1}{r^2}\partial_r\left(\rho\nu r^4\partial_r\Omega\right) + \underbrace{\frac{1}{5r^2}\partial_r\left(\rho r^4\Omega U\right)}_{\text{advection}} - \frac{1}{r^2}\partial_r\left(r^2F_J(r)\right)$$

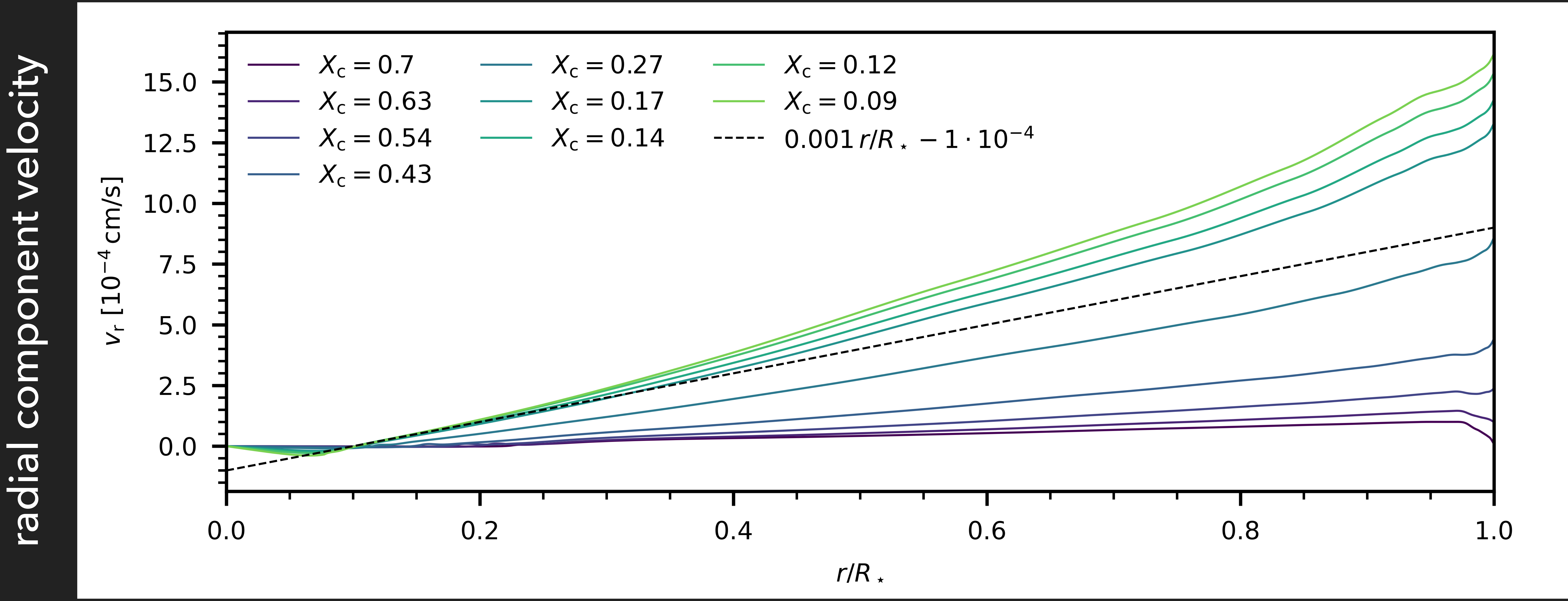
- Computing U is difficult. Codes that do advection (e.g. STAREVOL) use a parameterised expression from Maeder & Zahn 1998.

LAB 3: ADDING AN ADDITIONAL TORQUE - ADVECTION



Lead TA: Tryston Raecke

A VELOCITY PROFILE FROM ESTER



- Fit will not give a self-consistent velocity profile, so AM is not conserved.
- In 1D, we assume shellular rotation and study the average radial motion.
- In reality, the velocity scales with the viscosity.

OTHER HOOKS

```
subroutine default_other_torque(id, ierr)

   integer, intent(in) :: id
   integer, intent(out)  :: ierr
   type (star_info), pointer :: s
   ierr = 0
   call star_ptr(id, s, ierr)
   if (ierr /= 0) return

   s% extra_jdot(:) = 0
   s% extra_omegadot(:) = 0


end subroutine default_other_torque


subroutine extras_controls(id, ierr)
  …

  s% other_torque => default_other_torque


end subroutine extras_controls
```

```
&controls

use_other_torque = .true.
```

LAB 3: ADDING AN ADDITIONAL TORQUE - ADVECTION

- Add advection as an other torque

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\Omega\right) = \frac{1}{5r^2\rho}\partial_r\left(\rho r^4 \Omega U\right)$$

`s% extra_jdot`

- To conserve angular momentum the total torque should sum to zero

$$\tau_{\mathrm{tot}} = \sum_n \dot{j}_n \Delta m_n$$

radiative envelope

$$\dot{j}_n = -\frac{\tau_{\mathrm{tot}}}{(n_z - k_0)\Delta m_n}$$

convective core

SOME CODING TIPS

SOME CODING TIPS

- Define double precision variables as

```
real(dp) :: your_variable ! (and not double precision :: your_variable)
```

SOME CODING TIPS

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```
real(dp) :: your_variable ! (and not double precision :: your_variable)
```

- Numerical expressions as double

`37._dp` (or `37d0`)

SOME CODING TIPS

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real(dp) :: your_variable ! (and not double precision :: your_variable)
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- Exponents

`powN(x)`, or `pow(x, A)` (and not `x**N`)

SOME CODING TIPS

- Define double precision variables as

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real(dp) :: your_variable ! (and not double precision :: your_variable)
```

- Numerical expressions as double

`37._dp` (or `37d0`)

- Exponents

`powN(x)`, or `pow(x, A)` (and not `x**N`)

- Allocate array

```
real(dp), allocatable :: arr(:)
allocate(arr(N))
deallocate(arr) ! local arrays are automatically deallocated
```

CELL AND FACE QUANTITIES

- Examples of quantities defined at the face: m_k, r_k, L_k, v_k

- Examples of quantities defined at the cell: $\rho_k, T_k, X_{i,k}, P_k, \nabla_{\mathrm{ad},k}, \mathrm{d}m_k$

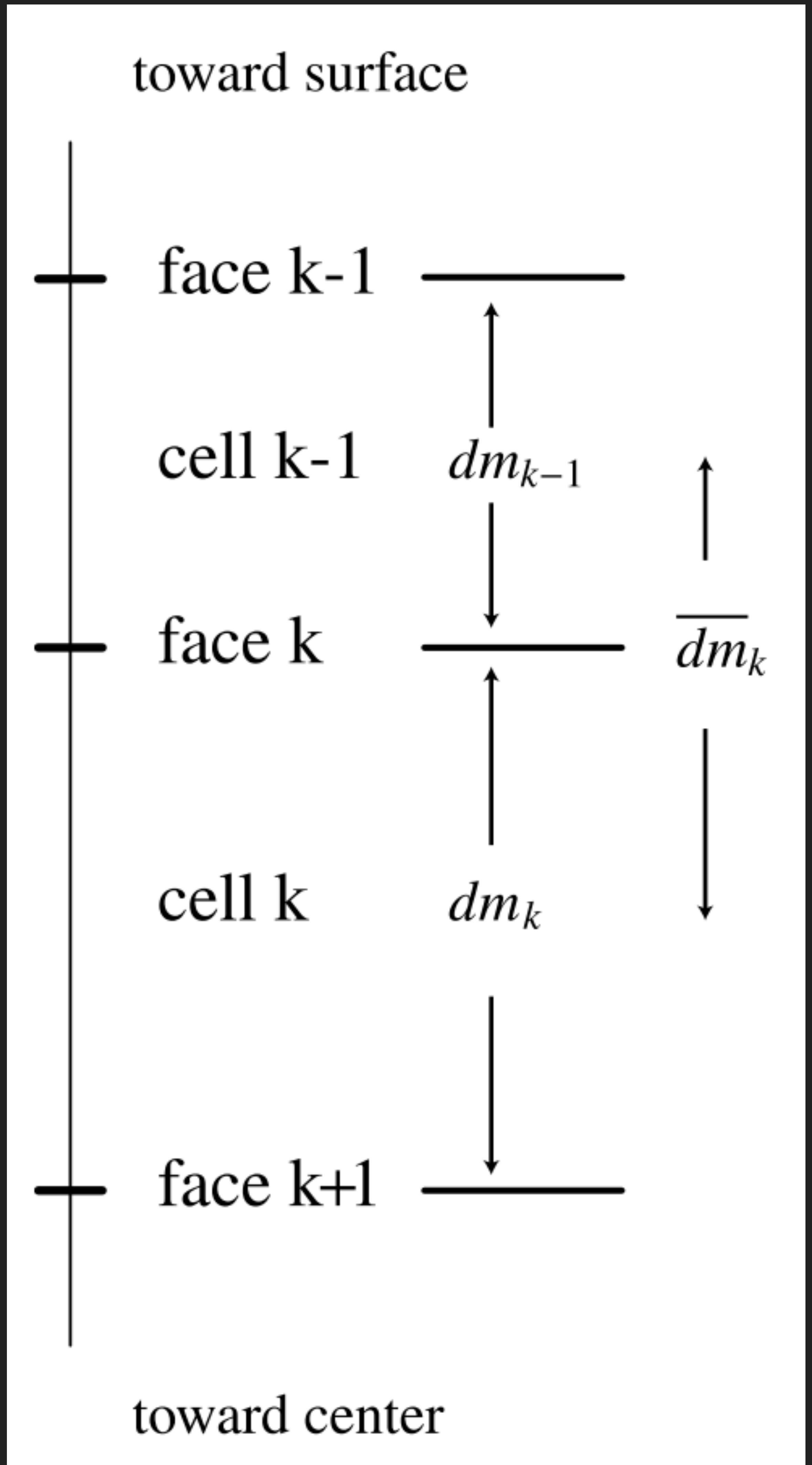

Fig. 9 in MESA Paper I, Paxton et al. (2011)

LAB 3 RECAP

- Meridional circulation can increase the differential rotation.
- In reality, the velocity evolves over time.