Lecture Notes on Asteroseismology for the 2025 MESA Summer School

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1 Introduction

Intuitively, if you strike a star hard enough, it will ring at certain characteristic frequencies; these are the frequencies of its normal modes of oscillation. The objective of asteroseismology — the analysis and interpretation of stellar oscillations — is to constrain a star's properties using these frequencies (as well as, in some cases, their amplitudes, phases, and damping rates).

Asteroseismology is simultaneously a rapidly growing discipline, but yet also one with an extremely long historical legacy dating back to the birth of astrophysics out of astronomy. The objective of the MESA Summer School is only to provide you with the skills of a numerical practitioner, rather than those of an observer or a theorist, so the discussion here will necessarily be only an incomplete overview of this field, its techniques, and its history. Nonetheless, to the extent that MESA and its ecosystem ought not to be a black box, these notes are intended to provide you with enough background knowledge to be able to interpret and potentially sanity-check your calculations, if necessary.

2 Some Theoretical Basics

Earlier in the Summer School, we have seen that MESA derives a numerical structure by solving a set of coupled differential equations. The specific ones that are also relevant to stellar oscillations are:

- The continuity equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0, \tag{1}$$
which specifies that mass is locally conserved;
- The momentum equation,
$$\rho \frac{\partial}{\partial t} \boldsymbol{v} = -\nabla P - \rho \nabla \Phi = \nabla P + \rho \mathbf{g}, \tag{2}$$
which relates body forces, such as hydrostatic pressure gradients and gravitation, to the velocity field $\boldsymbol{v}$;
- Laplace's equation for the gravitational potential,
$$\nabla^2 \Phi = 4\pi G \rho, \tag{3}$$

which relates the gravitational field to how mass is distributed, and

- The entropy equation,
$$\frac{\mathrm{d}S}{\mathrm{d}t} = \left.\frac{\partial S}{\partial \rho}\right|_P \frac{\mathrm{d}\rho}{\mathrm{d}t} + \left.\frac{\partial S}{\partial P}\right|_\rho \frac{\mathrm{d}P}{\mathrm{d}t}, \tag{4}$$
which relates the thermodynamic evolution of the star to that of its structure (through the density and pressure).

2.1 Linear Adiabatic Oscillations

Of primary interest to asteroseismologists are the mode frequencies that we would obtain for *linear, adiabatic* oscillations.

- *Linearity* means that we may perturb each of the quantities describing the equilibrium structure locally, as e.g. $\rho \to \rho + \lambda\rho'$, and ignore terms of second and higher order in λ. More broadly, it is what entitles us to consider any configuration of waves to be a linear combination of normal modes at different phases and amplitudes.
- *Adiabaticity* means that we may treat the oscillations as being so fast that we may as well consider the entropy not to vary over time, or indeed to vary at all, as a result of these oscillations.

We will moreover distinguish between an *Eulerian* description of perturbations to the fluid — in which we simply take the difference between scalar quantities at the same position, after vs. before the perturbation is applied — from a *Lagrangian* description, where we locally follow the flow of the fluid as the perturbation is applied. More precisely, at a position $\mathbf{x}$, the Eulerian perturbation is $\rho_{\text{after}}(\mathbf{x}) - \rho(\mathbf{x}) = \lambda\rho'(\mathbf{x}) + \mathcal{O}(\lambda^2)$. However, if the perturbation should also displace the fluid that originally was at $\mathbf{x}$ to a new position $\mathbf{x} + \lambda\boldsymbol{\xi}$, then the Lagrangian perturbation is

$$\rho_{\text{after}}(\mathbf{x}) - \rho(\mathbf{x} - \lambda\boldsymbol{\xi}) = \lambda\left(\rho'(\mathbf{x}) + \boldsymbol{\xi}\cdot\nabla\rho\right) + \mathcal{O}(\lambda^2) \equiv \lambda\delta\rho(\mathbf{x}). \tag{5}$$

Moreover, the equilibrium structure is assumed to be (on the timescales associated with stellar oscillations) invariant with respect to time, and so the velocity field itself can be expressed as the time derivative of the Lagrangian displacement function. Correspondingly, it is of order λ in smallness when keeping track of perturbations. In this language, we may thus write the leading-order perturbations to eqs. (1) to (4) as

$$\frac{\partial}{\partial t}\left(\rho' + \nabla\cdot\rho\boldsymbol{\xi}\right) = 0, \tag{Continuity}$$

$$\rho\frac{\partial^2}{\partial t^2}\boldsymbol{\xi} = -\nabla P' + \rho'\mathbf{g} - \rho\nabla\Phi', \tag{Momentum}$$

$$\nabla^2\Phi' = 4\pi G\rho', \tag{Gravitation}$$

$$\delta S = \left.\frac{\partial S}{\partial \rho}\right|_P \delta\rho + \left.\frac{\partial S}{\partial P}\right|_\rho \delta P = 0. \tag{Adiabaticity}$$

Let's rearrange all of this into a single equation. First, the adiabaticity constraint can be more compactly rewritten in terms of the adiabatic sound speed

$$-\left.\frac{\partial S}{\partial \rho}\right|_P \Big/ \left.\frac{\partial S}{\partial P}\right|_\rho = \left.\frac{\partial P}{\partial \rho}\right|_S = c_s^2, \tag{6}$$

so that

$$P' + \boldsymbol{\xi} \cdot \nabla P = P' + \boldsymbol{\xi} \cdot \rho \mathbf{g} = c_s^2 \left(\rho' + \boldsymbol{\xi} \cdot \nabla \rho\right). \tag{7}$$

Moreover, we will perform separation of variables, so that each of these perturbations is understood to be a function of only position, multiplied by a time-dependent term that goes as $e^{i\omega t}$, where ω is the angular frequency. Finally, the Laplacian operator in Laplace's equation can be inverted with the use of a Green's Function. Putting all of these together into the momentum equation yields

$$\begin{aligned} -\omega^2 \rho \boldsymbol{\xi} = &\nabla(\rho c_s^2 \nabla \cdot \boldsymbol{\xi}) + \nabla(\boldsymbol{\xi} \cdot \rho \mathbf{g}) - \rho \mathbf{g} \nabla \cdot \boldsymbol{\xi} - \mathbf{g}(\boldsymbol{\xi} \cdot \nabla \rho) \\ &- 4\pi G \rho \nabla \int \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \nabla \cdot (\rho(\mathbf{x}')\boldsymbol{\xi}(\mathbf{x}'))\, \mathrm{d}^3 x'. \end{aligned} \tag{8}$$

Each of the differently-coloured blocks on the RHS is, separately, self-adjoint in the following sense. Let's say $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are two square-integrable Lagrangian displacement functions. For the first term, applying the divergence theorem yields

$$\int \boldsymbol{\xi}_1^* \cdot \nabla(\rho c_s^2 \nabla \cdot \boldsymbol{\xi}_2) \mathrm{d}^3 x = \left(\int \boldsymbol{\xi}_2^* \cdot \nabla(\rho c_s^2 \nabla \cdot \boldsymbol{\xi}_1) \mathrm{d}^3 x\right)^*, \tag{9}$$

subject to the integral over the boundary of the domain vanishing. This is true for each of the coloured blocks separately, so eq. (8) describes a global *eigenvalue problem* generated by a self-adjoint linear operator (and accompanying boundary conditions), with eigenvalues $-\omega^2$. Since the problem is self-adjoint, these eigenvalues must be real. If you're familiar with differential equations, this is similar to, but not quite the same as, a Sturm-Liouville problem.

Before we approach the problem of solving for normal modes (a global problem), let's first examine the qualitative behaviour of this operator in a local sense. In particular, let's employ the approximation that these waves can be described with a local wavevector $\mathbf{k}$, such that $\nabla \cdot \boldsymbol{\xi} = i\mathbf{k} \cdot \boldsymbol{\xi}$. This is analogous to JWKB analysis in classical or quantum mechanics, and strictly speaking is only approximately accurate when the components of k are large. Using this, eq. (8) can be rewritten as

$$-\omega^2 \rho \boldsymbol{\xi} = \left(-c_s^2 \mathbf{k}\mathbf{k}^T + i(\mathbf{k}\mathbf{g}^T - \mathbf{g}\mathbf{k}^T) + \frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\mathbf{e_r}\mathbf{e_r}^T - \frac{4\pi G \rho}{|\mathbf{k}|^2}\left(\mathbf{k} + \frac{i}{\rho}\nabla\rho\right)\left(\mathbf{k} - \frac{i}{\rho}\nabla\rho\right)^T\right)\rho\boldsymbol{\xi}, \tag{10}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction. Each of the coloured blocks are now Hermitian matrices, giving us a *local* eigenvalue problem. Since the frequency eigenvalues are fixed (a wave's frequency doesn't change with position), solving for them in terms of $\mathbf{k}$ gives us a *dispersion relation*, and the different eigenvectors of this matrix specify different *polarisation states*.

Let's consider what kind of dispersion relations we might obtain. In general we have three polarisation states in three dimensions, but since most stars are spherically symmetric we may consider only fluid motions in the radial and one notional horizontal direction. The red term alone is the only term we would obtain in a homogenous, isotropic, and nongravitating medium, and it has only one non-null eigenvector, which is $\mathbf{k}$ itself. This corresponds to a *longitudinal* polarisation, where the fluid displacement is along the same direction as the wave itself travels. For this eigenvector, $\omega = c_s k$, with no dependence on the direction of $\mathbf{k}$. These are *acoustic* waves: they propagate at the same speed in all directions, and propagate longitudinally. Pressure support provides the restoring force for these waves, so they are referred to as p-waves (or p-modes when they form normal modes). The other eigenvectors, representing transverse waves, must be orthogonal to this, since the matrix is Hermitian; they are annihilated by this matrix, and so each have eigenvalue 0.

Likewise, the blue term alone is what we would obtain if we were to ignore density stratification, and also demand that the wavevector's components satisfy $k_r^2 + k_h^2 = 0$. For example, we could demand that $k_r = \pm i k_h$ be imaginary, so that that $\boldsymbol{\xi} \sim e^{\pm k_h r}$ — it decays away from the centre, or (more usually) away from the surface. In that case, we have that the frequency eigenvalues are $\omega^2 = \pm g k_h$, with eigenvectors $\boldsymbol{\xi} \sim i\mathbf{e}_r \pm \mathbf{e}_h$ — the displacement in the radial direction is phase-lagged from that in the horizontal direction by $\pm\frac{\pi}{2}$. These are horizontal surface gravity waves (whose quantised normal modes are sometimes referred to as 'f-modes').

Let's expand our consideration now to include both the terms in red (encoding the behaviour of acoustic waves), and those in blue and orange (which are proportional to the local gravitational field strength), but ignore the term in black, which depends on the perturbation to the gravitational potential. That is, the wave feels the effect of gravity, but approximately does not itself affect the gravitational field. This is called **Cowling's approximation**. With respect to radial and horizontal displacements, eq. (10) may be written

$$\omega^2 \rho \begin{bmatrix} \xi_r \\ \xi_h \end{bmatrix} = \begin{bmatrix} c_s^2 k_r^2 - \frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} & c_s^2 k_r k_h + i g k_h \\ c_s^2 k_r k_h - i g k_h & c_s^2 k_h^2 \end{bmatrix} \rho \begin{bmatrix} \xi_r \\ \xi_h \end{bmatrix} \equiv \rho \mathbf{L} \boldsymbol{\xi}. \tag{11}$$

We would find frequency eigenvalues from this expression by solving for the roots of the characteristic polynomial,

$$\det\left[\mathbf{L} - \omega^2 \mathbb{I}_2\right] = \left(c_s^2 k_r^2 - \frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \omega^2\right)\left(c_s^2 k_h^2 - \omega^2\right) - \left((c_s^2 k_r k_h)^2 + (g k_h)^2\right) = 0. \tag{12}$$

Regrouping terms, we may rewrite this as

$$\begin{aligned} c_s^2 k_r^2 &= \omega^2 \left[\left(1 - \frac{c_s^2 k_h^2}{\omega^2}\right)\left(1 + \frac{1}{\omega^2}\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) - \left(\frac{g k_h}{\omega^2}\right)^2\right] \\ &= \omega^2 \left[\left(1 - \frac{c_s^2 k_h^2}{\omega^2}\right)\left(1 - \frac{1}{\omega^2}\left(-\frac{g^2}{c_s^2} - \frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)\right) - \frac{g^2}{c_s^2 \omega^2}\right] \\ &\equiv \omega^2 \left(1 - \frac{c_s^2 k_h^2}{\omega^2}\right)\left(1 - \frac{N^2}{\omega^2}\right) - \left(\frac{c_s}{\Gamma_1 H_p}\right)^2. \end{aligned} \tag{13}$$

Here we have introduced the Brunt-Väisälä frequency

$$N^2 = g\left(\frac{1}{\Gamma_1 P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) \tag{14}$$

as well as the pressure scale height $H_p = \left(-\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\right)^{-1}$. Ignoring the final term in eq. (13) for the moment, this gives us a discriminant for the behaviour of the wave at any given location:

- If both $\omega^2 > c_s^2 k_h^2$ and $\omega^2 > N^2$, then k_r is real, and the wave is oscillatory. This is also true if both $\omega^2 < c_s^2 k_h^2$ and $\omega^2 < N^2$.
- Otherwise, k_r is imaginary, and the wave is evanescent.

Both of these statements are still true, albeit in a modified form, when the final term is included — it is an "acoustic cutoff" frequency which only becomes significant in the near-surface layers. Since the terms appearing in square brackets are a quadratic polynomial in ω^2, we can always factorise them into the form $(1 - \omega_+^2/\omega^2)(1 - \omega_-^2/\omega^2)$, where $\omega_\pm^2$ are the roots of the polynomial. However, these discriminating frequencies are well approximated by N^2 and $c_s^2 k_h^2$, particularly far away from the surface.

Equation (13) itself is a quadratic polynomial in ω^2, and it can be factorised to yield

$$\omega^2 = \left[\frac{c_s^2(k_r^2 + k_h^2) + N^2 + \frac{c_s^2}{\Gamma_1^2 H_p^2}}{2}\right] \pm \sqrt{[\ldots]^2 + c_s^2 k_h^2 N^2}, \tag{15}$$

where the ellipses repeat the term in the square brackets outside the square root. These two different eigenvalues correspond to different kinds of waves, which, we have established, can only propagate at either high or low frequencies. The positive branch corresponds to high frequencies, where

$$\omega_p^2 \sim c_s^2(k_r^2 + k_h^2) + N^2 + \frac{c_s^2}{\Gamma_1^2 H_p^2} \sim c_s^2(k_r^2 + k_h^2), \tag{16}$$

in the limit of a wave with very short wavelength. By inspection, this reduces to the isotropic acoustic-wave dispersion relation that we saw earlier, which is polarised longitudinally as $\boldsymbol{\xi} \sim \mathbf{k}$. By contrast, the negative branch corresponds to low frequencies, where

$$\omega_g^2 \sim \frac{k_h^2 N^2}{k_r^2 + k_h^2}, \tag{17}$$

favouring horizontal propagation. These are *buoyancy* or *gravity* waves, yielding g-modes. Nontrivially, gravity waves are polarised such that

$$\boldsymbol{\xi} \sim \begin{bmatrix} -k_h \left(1 - \frac{N^2}{c_s^2 k^2}\right) \\ k_r \left(1 - i\frac{g}{c_s^2 k_r}\right) \end{bmatrix}; \tag{18}$$

that is to say, the wave is largely a *transverse* wave (with the fluid displacement being mostly perpendicular to the direction of travel), and the horizontal component is moreover phase-lagged from the radial component by a small amount.

2.2 Rotation

In order to proceed further, we must now distinguish between the mode frequency as felt by a wave in a co-rotating coordinate system, and one seen by an observer in an inertial reference frame. We will denote mode frequencies in the co-rotating coordinate system with σ rather than ω, and reserve ω for mode frequencies seen by an inertial observer.

In the corotating frame, rotation modifies the wave operator of eq. (8) by introducing, to leading order in the rotation rate Ω, an additional term describing the effect of the Coriolis force:

$$-2\rho\boldsymbol{\Omega} \times \mathbf{v} = -2i\rho\sigma\boldsymbol{\Omega} \times \boldsymbol{\xi}. \tag{19}$$

This term is manifestly Hermitian (as can be proven solely from vector calculus identities). For our subsequent discussion it will be convenient to set up our corotating coordinate system such that $\boldsymbol{\Omega}$ points along the positive z-axis.

Again, it is helpful to think first of the eigensystem of this operator in isolation (i.e. ignoring pressure and buoyancy support), before considering how it might modify the dispersion relation above. The former case is a good approximation when the rotation rate is comparable to, or much faster than, the corotating mode frequency. Now, no spatial derivatives appear for this term isolation, so its contribution to the dispersion relation prima facie does not appear to depend on the wavevector $\mathbf{k}$. However, we may remedy this by taking the curl of both sides, which becomes a simple vector cross product $i\mathbf{k}\times$ in our "JWKB" approximation, yielding

$$-i\sigma\mathbf{k}\times\rho\boldsymbol{\xi} = -2\mathbf{k}\times(\boldsymbol{\Omega}\times\rho\boldsymbol{\xi}) = -2(\boldsymbol{\Omega}(\mathbf{k}\cdot\rho\boldsymbol{\xi}) - \rho\boldsymbol{\xi}(\boldsymbol{\Omega}\cdot\mathbf{k})). \tag{20}$$

This would be a *Generalised Hermitian eigenvalue problem* in the frequency eigenvalue σ, since the linear operator acting on $\rho\boldsymbol{\xi}$ on the left-hand-side, proportional to σ, is Hermitian. However, by inspection, the first term on the right-hand-side is not Hermitian for arbitrary $\mathbf{k}$ and $\boldsymbol{\Omega}$: it is so *only if* either $\mathbf{k}$ is parallel to $\boldsymbol{\Omega}$ everywhere, or else if $\rho\boldsymbol{\xi}$ is perpendicular to $\mathbf{k}$ (so that the term vanishes). This gives us two classes of solutions, but it turns out that the one with $\mathbf{k}$ pointing along the z-axis everywhere is trivial. The other class of solutions has frequency eigenvalues σ such that

$$\sigma = \pm 2\frac{\boldsymbol{\Omega}\cdot\mathbf{k}}{|\mathbf{k}|}, \tag{21}$$

where the eigenvectors $\rho\boldsymbol{\xi}$ are *circularly polarised*: they rotate at this frequency around an axis defined by $\mathbf{k}$. In particular, this means that the frequency of such propagating waves is bounded strictly from above:

$$|\sigma| \leq 2\Omega. \tag{22}$$

Waves of this kind are called 'inertial' waves. The normal modes formed by these waves are notorious for being very difficult to describe using the spherically-symmetric formulation (described below) that is otherwise commonly used for pulsation calculations.

A full analysis of how rotation might interact with pressure and buoyancy support lies outside the scope of these notes. For one, it also ought to take into account the deformation to the equilibrium structure that would result from the centrifugal force (a second-order rotational effect). Additionally, since vector cross products are only well defined in three (but not two) dimensions, this means that we now also in general must account for both possible horizontal directions, rather than just one as in our earlier analysis. However, matters are greatly simplified if we are to use what is often called the **Traditional Approximation to Rotation**, or the TAR for short. This is where the rotational vector $\boldsymbol{\Omega}$ is replaced with $\boldsymbol{\Omega}_r = (\boldsymbol{\Omega}\cdot\mathbf{e}_r)\mathbf{e}_r = \Omega\cos\theta\,\mathbf{e}_r$. As a result, we have both that $\boldsymbol{\Omega}_r\times\boldsymbol{\xi}_r = 0$, and that $\mathbf{e}_r\cdot\boldsymbol{\Omega}_r\times\boldsymbol{\xi} = 0$ — only the horizontal components of the pulsation eigenfunction couple directly to the Coriolis force under this approximation. The matrix representation of the dispersion relation can then easily be written in block matrix form as

$$\sigma^2\rho\begin{bmatrix}\xi_r\\ \boldsymbol{\xi}_h\end{bmatrix} = \begin{bmatrix} c_s^2k_r^2 - \frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} & c_s^2k_r\mathbf{k}_h^T + ig\mathbf{k}_h^T \\ c_s^2k_r\mathbf{k}_h - ig\mathbf{k}_h & c_s^2\mathbf{k}_h\mathbf{k}_h^T + 2\sigma\Omega_r\mathbf{J}_r\end{bmatrix}\rho\begin{bmatrix}\xi_r\\ \boldsymbol{\xi}_h\end{bmatrix}, \tag{23}$$

where $\mathbf{k}_h$ and $\boldsymbol{\xi}_h$ are vectors perpendicular to the radial direction, and we have represented the cross product $\boldsymbol{\Omega}_r\times\boldsymbol{\xi}_h$ as a linear operator in some notional right-handed coordinate basis using the unit angular momentum matrix $\mathbf{J}_r = i\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, which is invariant under 2D rotations. The frequency eigenvalue σ now appears on both sides of this equation. This is a *Quadratic Hermitian eigenvalue problem* — we will not be analysing it in detail in these notes.

2.3 Coordinate Formulation

The numerical tools that you will use today solve the same eigenvalue problem as eq. (8). Rather than using it directly an integro-differential equation, these standard tools solve a set of coupled ordinary differential equations, obtained by applying separation of variables to eqs. (Continuity) to (Gravitation) with the dependence on the horizontal coordinates specified by the spherical harmonics in the absence of rotation. Recall that the spherical harmonic functions Y_ℓ^m satisfy

$$\begin{aligned} \left(\frac{\partial^2}{\partial\phi^2} + m^2\right) Y_\ell^m = 0, \text{ and} \\ \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \ell\left(\ell+1\right)\right) Y_\ell^m \equiv \left(r^2\nabla_h^2 + \ell(\ell+1)\right) Y_\ell^m = 0. \end{aligned} \tag{24}$$

For spherical harmonics with degree ℓ and azimuthal order m, this means by definition that

$$k_h^2 = \frac{\ell(\ell+1)}{r^2}, \tag{25}$$

which is exact rather than approximate. Inserting this into those equations, and explicitly expressing the radial dependence of all functions in spherical coordinates, gives at last that

$$\begin{aligned} \frac{\mathrm{d}\xi_r}{\mathrm{d}r} &= \left(\frac{g}{c_s^2} - \frac{2}{r}\right)\xi_r - \left(1 - \frac{\ell(\ell+1)c_s^2}{r^2\omega^2}\right)\frac{P'}{\rho c_s^2} + \frac{\ell(\ell+1)}{r^2\omega^2}\Phi', \\ \frac{\mathrm{d}P'}{\mathrm{d}r} &= \rho(\omega^2 - N^2)\xi_r - \frac{g}{c_s^2}P' - \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r}, \\ \frac{\mathrm{d}\Phi'}{\mathrm{d}r} &= \frac{\mathrm{d}\Phi'}{\mathrm{d}r}, \text{ and} \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) &= \frac{4\pi G\rho N^2}{g}\xi_r + \frac{4\pi G}{c_s^2}P' + \frac{\ell(\ell+1)}{r^2}\Phi' - \frac{2}{r}\frac{\mathrm{d}\Phi'}{\mathrm{d}r}. \end{aligned} \tag{26}$$

The numerical schemes used by `GYRE` and `ADIPLS` — the two pulsation solvers that are bundled with `MESA` — are expressed either in these variables, or alternatively using the horizontal displacement function $\xi_h = (P' + \rho\Phi')/r\omega^2$ in lieu of the Eulerian pressure perturbation. These two codes attempt to find solutions to these differential equations that satisfy two boundary conditions each at the center and at the surface of the mesh. This is only possible at certain values of the frequency ω, which are returned as the normal-mode frequency eigenvalues. The eigenfunctions associated with each such eigenvalue form an orthogonal basis with respect to the inner product

$$\int \rho\boldsymbol{\xi}_i^* \cdot \boldsymbol{\xi}_j \mathrm{d}^3x = \int r^2\rho\left(\xi_{r,i}^*\xi_{r,j} + \ell(\ell+1)\xi_{h,i}^*\xi_{h,j}\right)\mathrm{d}r \propto \delta_{ij}, \tag{27}$$

for modes of the same ℓ.

In the presence of rotation, something interesting happens. In the rotating reference frame, the elements of $\boldsymbol{\xi}$ pointing in the θ and ϕ directions satisfy

$$\left(-\sigma^2\mathbb{I} - 2\sigma\Omega_r\mathbf{J}_r\right)\rho\boldsymbol{\xi}_h = -\nabla_h P', \tag{28}$$

when we use both the Cowling Approximation and the Traditional Approximation to Rotation. Now, under separation of variables as $\rho' \to \tilde{\rho}'(r)F(\theta,\phi)$, the continuity equation $\rho' + \nabla\cdot\rho\boldsymbol{\xi}$ yields a constraint that $\nabla_h\cdot\boldsymbol{\xi}_h =$ some constant. This gives us that

$$\nabla_h\cdot\left(\sigma^2\mathbb{I} + 2\sigma\Omega_r\mathbf{J}_r\right)^{-1}\nabla_h F + \sigma^2\lambda F = 0, \tag{29}$$

for some constant eigenvalue λ. Defining $q = 2\Omega/\sigma$ and $\mu = \cos\theta$, and allowing ourselves azimuthal symmetry so that $\frac{\partial}{\partial\phi} \to im$, one obtains at last that

$$\left[\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{1-\mu^2}{1-q^2\mu^2}\frac{\mathrm{d}}{\mathrm{d}\mu}\right) - \frac{m^2}{(1-\mu^2)(1-q^2\mu^2)} + \frac{mq(1+q^2\mu^2)}{(1-q^2\mu^2)^2}\right] F + \lambda F = 0. \quad (30)$$

This is a differential equation of Sturm-Liouville form, and when $q \to 0$ it reduces to the second line of eq. (24). That is to say, under the Traditional Approximation to Rotation (in conjunction with Cowling's Approximation), separation of variables is performed with respect to not the spherical harmonics, which satisfy the Laplace Equation eq. (24), but rather horizontal eigenfunctions which satisfy eq. (30) — this is often referred to as Laplace's Tidal Equation. The eigenfunctions F are known as the Hough functions. There are two families of tidal eigenvalues λ — one family is quantised that such that each $\lambda \to \ell(\ell+1)$ as $q \to 0$, and the other (the "Rossby" family) goes to 0 in the same limit.

2.4 Perturbation Theory

Equation (8) is an operator eigenvalue equation of the generic form

$$\hat{\mathcal{L}}\left|\boldsymbol{\xi}\right\rangle = -\rho\omega^2\left|\boldsymbol{\xi}\right\rangle, \quad (31)$$

where, analogously to quantum mechanics, the eigenfunction $\boldsymbol{\xi}(x)$ is found by projecting the state vector into the position basis as $\boldsymbol{\xi}(\mathbf{x}) = \langle\mathbf{x}|\boldsymbol{\xi}\rangle$. This being so, small perturbations to the wave operator, of the form $\hat{\mathcal{L}} \mapsto \hat{\mathcal{L}} + \epsilon\hat{\mathcal{V}}$, can be related to small perturbations to the frequency eigenvalues as expanded in powers of ϵ, e.g. through the use of Rayleigh-Schrödinger perturbation theory.

We are concerned primarily with rotation as a dynamical perturbation today, and so the perturbation to the wave operator is of the specific form

$$V_{jk} = \left\langle\boldsymbol{\xi}_j|\hat{\mathcal{V}}|\boldsymbol{\xi}_k\right\rangle = -2i\sigma\int\rho\left(\boldsymbol{\xi}_j^*\cdot\boldsymbol{\Omega}\times\boldsymbol{\xi}_k\right)\mathrm{d}^3x \quad (32)$$

in the corotating frame, as follows from eq. (19). Upon transformation to the stationary frame and projection onto the basis of spherical harmonics, this has the coordinate representation

$$R_{ij} = 2m\int\mathrm{d}r\,\Omega(r)\,r^2\rho_0\left(\xi_{r,i}\xi_{r,j} + [\ell(\ell+1)-1]\xi_{t,i}\xi_{t,j} - \xi_{r,i}\xi_{t,j} - \xi_{t,i}\xi_{r,j}\right) \equiv 2m\beta_{ij}\int\mathrm{d}r\,\Omega(r)K_{ij}(r), \quad (33)$$

where each kernel function K_{ij} is defined to have unit integral, so that the overall sensitivity of the matrix element to solid-body rotation is carried by the sensitivity constant β_{ij}.

To first order in perturbation theory, the effect of such a perturbation to the wave operator is specified by the diagonal elements of the corresponding perturbation matrices. For rotation specifically, the i^{th} mode is shifted in frequency by $R_{ii}/2$ under the action of rotation. In the case of solid-body rotation, this means that each mode experiences a shift

$$\omega_i \mapsto \omega_i + m\beta_i\Omega_i \equiv \omega_i + m\delta\omega_i; \quad (34)$$

since modes of the same n but different ℓ and m would pulsate at the same frequency in a nonrotating star, rotation has the effect of turning a single mode into a multiplet of modes (with $2\ell+1$ components in each multiplet).

2.5 Further Reading

We recommend the following reference texts to supplement the description above:

- *Nonradial Oscillations of Stars*, W. Unno, Y. Osaki, H. Ando, H. Saio, & H. Shibahashi, 1989 (Tokyo: Univ. Tokyo Press)
- *Linear Adiabatic Stellar Oscillations*, D. O. Gough, 1993, in Astrophysical Fluid Dynamics — Les Houches 1987, ed. J.-P. Zahn & J. Zinn-Justin (Amsterdam: North-Holland), p. 399
- *Lecture Notes on Stellar Oscillations*, J. Christensen-Dalsgaard, 2008
- *Asteroseismology*, C. Aerts, J. Christensen-Dalsgaard, & D. Kurtz, 2010 (Berlin: Springer)